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Nonlinear Adaptive Control
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Week - 5,
Lecture 25
Exponential Stability and Converse Lyapunov Theorems

Hello everyone, welcome to yet another week of our NPTEL on Non-linear and Adaptive Control. I am Srikant Sukumar from Systems and Control IIT, Bombay. So, we are just entering our fifth week of this course and just to commemorate the finishing of four weeks or one third of what this course is supposed to be I have changed our background to a very nice spacex satellite which is orbiting the earth.

Until now we had this nice background of a rover on mars and we said that you use these autonomous algorithms to determine where the rover is or its position and where it is supposed to go and there are algorithms that drive the rover in a particular path, because it cannot necessarily be remote controlled in real time from the earth.

Similar situation is that for spacecraft also, a lot of what we do in adaptive control has to some extent been implemented in by the spacecraft community, by the satellite community. And so, in order to maintain both the position of the spacecraft in orbit and also its orientation in orbit which may be a rather mission-critical component we at several occasions require the use of adaptive control because it is not very easy to figure out several space parameters like the inertia, mass, etc of the spacecraft which are also of course losing fuel at a pretty good rate as they are flying.

So anyway, so this is another motivating application for us, in fact, a lot of my own work in my Ph.D. in Aerospace Engineering was in fact using adaptive and non-linear control to spacecraft problems. So, anyway so hopefully again we will continue to see this background for a while and hopefully we can motivate ourselves to design and develop algorithms that can drive systems such as this spacecraft, excellent. So, what are we doing until now?

So, we were actually somewhere here, we had not finished the lecture notes from previous week in fact, and this is why I had mentioned that we should not worry too much if we are sort of moving ahead in the lecture notes because I knew that as we move forward the material will get more involved and we will start to get a little bit slower.

So, but by now I think most of you understand what is the notion, what are the notions of stability what is the pitfalls in analysis, how to sort of we had a grip of how to use the Barbalat's lemma, we also saw are now looking at the Lyapunov stability theorems. So, we have already looked at the stability theorems until global uniform asymptotic stability.

So, now we are left with only the exponential stability theorems. So, as you all might recall exponential stability essentially is a more advanced version of asymptotic stability I would say and why is it more advanced, it is so because it actually gives you a particular rate of convergence to the equilibrium, which is assumed to be the origin in almost all our problems. So, that is the idea of exponential stability. So, the question is what do we require, what do we require for exponential stability?

So, we already have our usual assumptions that we need to have a candidate Lyapunov function which means that it is a C1 function which is positive definite, so this is the least that we do require. Beyond that we require V to be decrescent because these are the properties that you already have already started to see appearing, so we require V to be decrescent because exponential stability is by definition also uniform. So, V is required to be decrescent.

Further we require three class K functions all of the same order of magnitude, we will talk about what this means soon enough, same order of magnitude, we will talk about this soon enough, let us not worry too much about what this means. But we require three class K functions such that V is bounded on both ends in this way.

In fact, although I have mentioned V is decresent but if you look at this particular, if you look at in fact let me be careful here, this is not really required, there is no time argument here, there is no time argument here. The third one is fine but here there is no time argument. So, what we, if we look at this left-hand side this, what does this mean? This means that V is positive definite, this is requiring V is to be positive definite.

If you look at the right side inequality, this require, this implies that V is decrescent, so this is not actually required. So, if I, I can sort of say this, this is imply and this guy this implies in fact I should be careful, I should say it is a one-way straight for, this implies that this is positive definite.

And further the third requirement is something like this that V dot is less than equal to some negative of class K function. Now, this again, so here also of course we do not need this time argument, let us be careful, time argument is not needed, V dot is what we have defined already, it is the Lie derivative or the directional derivative of V.

And this particular piece, this means what? This already means implies I would say, not means but this implies negative definiteness of V, of V dot sorry. So, if you look at it, it seems like everything that we require here is already been mentioned here. Well I am sorry everything that is required here we got looking at the local version is already been mentioned here that V being positive definite is part of the candidate Lyapunov function definition, V being decrescent is mentioned here and V dot being negative definite is mentioned here.

So, it looks like the conditions are exactly the same, but there is a very nice and subtle little difference here and that is these words same order of magnitude this is the difference, same order of magnitude. So, here we did require class K functions to claim all of these but we did not say anything about them being the same order but for local exponential stability we do require them to be of the same order of magnitude. So, this is important, let us try to remember this.

Now, the global version of course if it moves from V being positive definite to V being radially unbounded and of course you will also require that V be valid over the entire domain Rn unlike what is specified up top, so we required this also, and we required this also, that V be valid or V be defined in the entire domain. And of course, we need now class KR functions, three class KR functions all of the same order of magnitude.

Such that you have the same type of inequalities, I mean I will again remove these when you are doing this comparison there is no time, because we are not, when we do these comparisons we are not looking at x as a solution of a dynamical system but x as just a variable, so it is not appropriate to put a time argument in here.

And so, if this happens so if you have radial unboundedness for V and you have three class KR functions then you of course you get something like global exponential stability. So, this is the notions of local and global exponential stability. So, let us sort of try to see what means for functions to be same order of magnitude.

So, we say the two functions in reals so here we are talking about fy and gy mapping to the real numbers. They are said to be the same order of magnitude if they are comparable using scalar quantities. That is if there exists some scalars gamma 1, gamma 2 positive such that g is bounded on both sides by f scaled by these gammas 1s and gamma 2s and vice versa, and vice versa.

Because if you see if this happens then the flipped version can also happen, because from here I can also say that, so this also implies if you notice that I can write fy bounded on both sides 1 over gamma 1 g y and something like 1 over gamma 2 g y, so this is equivalent, these are equivalent. So, if basically one function can be bounded on both sides by the other function then the converse is also true, should make sense.

So, two functions are said to be of the same order of magnitude if they can be compared with each other by just scalar quantity, scalar constants, yes, scalar constants, not functions of state, not functions of time, scalar constants so for all the above theorems so this is what it means to be of same order of magnitude.

So, for all the above theorems, so this is what it means for the same order of magnitude. We say that if we satisfy any of these theorems it is said to be the Lyapunov function then that particular V is said to be a Lyapunov function, if it satisfies any one of these. So, any of one of these bullet points, many bullet points then it is said to be a Lyapunov function, not a candidate Lyapunov function like here, but a Lyapunov function.

So, although we do not talk about it but they do exist converse Lyapunov functions and the statements always go something like this, if x equal to 0 is stable for some system star then there exists V t, x positive definite such that, positive definite and C1 such that something happens. So, this will, this is what the converse statements look like. The converse statement essentially say that if your system is stable or asymptotically stable or exponentially stable then you are guaranteed to have V with these kind of properties with the with the properties that we are talking about, there it is, it satisfies one of these bullet points.

So, they do exist converse theorems, they do exist converse theorem but of course it is not a constructive evidence. So, just the existence of a converse theorem does not mean you can find a Lyapunov function, it is not always easy to find such a Lyapunov function, just using the fact that there exists a converse theorem. So, this is the unfortunate truth.

So, that is why we do not, we are not really stressing too much on it or actually stating these theorems. So, there is an example that we want to do now but before we move on to the example I want us to look back at the previous examples and see if there was any system which was already exponentially stable.

So, we did a few examples, if you look at let us see did we do a, so here, no this was a harmonic oscillator which was just stable then we did I mean we could not talk about stability because in fact unstable then we found a new non-uniform stable example then we did this example this is the damped harmonic oscillatory, this is a damped harmonic oscillator.

And what did we actually do we took a Lyapunov function which is something like this which is K x 1 plus x 2 squared by 2 and x 1 squared by 2 alpha and we could show that V dot is also the same in some sense, V dot also has K x 1 plus x 2 square and x 1 square, it has the same terms as this guy, K x 1 plus x 2 square and x 1 square.

And from here we claimed global uniform asymptotic stability. But the fact is that this is also globally exponentially stable. Why? So, the first thing is that V is positive definite it satisfies all the properties here, so in fact V is radially unbounded not positive definite, V is radially unbounded and so V is lower bounded by a class K r function.

Similarly, V dot is also radially unbounded because it has the same terms as this, if this is relatively unbounded, V is radially bonded and V dot is also radially unbounded. Therefore, V dot also is upper bounded by a class K function because it is a negative sign, so it is upper bounded by class K function.

And of course, decrescence is free because there is no time argument. So, therefore, it is also upper bound, V is also upper bounded by class K function, so V is both upper and lower bounded by a class K r function and V dot is upper bounded by the negative of a class K r function. So, this is one of the things. The next thing is same order of magnitude of these functions, same order of magnitude. So, what are these class K r functions in fact? So, if I look at what these class K r functions will be in fact I will take exactly this function itself as my class K r function.

So, if you notice I had this phi 1, phi 2, phi 3 in class K r. So, here I will take my if I may let me make it bigger, I will take my, let me write in some other colour, actually I will take my phi 1 as half K x 1 plus x 2 square plus 1 over 2 alpha x 2 squared which is exactly V, sorry this is x 1 square, so this is also the same as phi 2 and this is the same as V because I can use the same on both sides.

And phi 3 I will use as this guy minus 1 minus k k x 1 plus x 2 whole square no no minus actually, not in the class K function plus K over alpha x 1 square, this is what I take as my phi 3. And if you look at phi 1, phi 2, phi 3 first of all, these are radially unbounded functions, this is not difficult to verify, they are positive definite, how are they positive definite, we have already verified this is positive definite earlier so I do not really need to do anything because this was already done, I do not want to repeat it, and you can see from here.

Now, radial unboundedness is obvious because if x 1 and x 2 go to infinity in any direction the only problematic direction would have been, just a second, only problematic direction would have been when K x 1 plus x 2 is 0. But in that case also x 1 is going to infinity still, because x 1 and x 2 both have to go to infinity in some direction.

So, the only problematic direction would have been where this is 0 but then x 1 is still going to infinity so this whole thing is going to infinity. So, these are in fact class K r functions no questions asked. Now, the fact that they are same order of magnitude is also very straightforward to verify, you can see that they are just related by some constants, just different constants, these constants are the only things that are different.

So, it is very easy for me to relate phi 1, phi 2 and phi 3 by some, in fact phi 1 is equal to phi 2, so the constants are 1, for, to relate phi 1, phi 2 and phi 3 I just need to find another constant which is very easy because they are just multiplied by some constants, so that is it. So, this is in fact turns out to be globally exponentially stable by this argument. So, we have already seen an example of global exponential stability.

Now, if you look at this guy on the other hand like this pendulum or a variation of the pendulum example in fact, if you look at this example it is uniform asymptotically so but it is not even globally uniformly asymptotically stable because this is not a radially unbounded V. So, the question is it locally exponentially stable at least, is it at least locally exponentially stable would be the question.

This is not very easy to verify, this is not very easy to verify. Because what we have to do is now compare this guy and this guy, compare this and this. Now, comparing this and this is not easy because if you look at actually we are going to compare the particular class K r, K functions that bound these but even if I try to compare these two quantities see x2 is the same, it is the same function just with different gains so this is not a problem.

But this one could be a problem, this one could be a problem because at x 1 equal to I mean, so say at x 1 equal to pi what happens let us see what happens at x 1 equal to pi what happens actually we are not including pi. So, we might still have some chance but this is not very obvious a case of a same order class K functions.

In fact, there is a trigonometric formula for 1 minus cos x 1 which is I think sine square x 1 by 2 I believe I think it is sine square x 1 by 2. I will have to verify this though, but I believe it is sine square x 1 by 2 but here I have sine square x 1. So, we still have to see if these two actually have some kind of an exponential connection with each other, sorry, same order or a scalar connection with each other.

And this is not super easy to verify I can tell you that. So, it is not very clear if this is in fact locally exponentially stable or not for almost certainly it is not, but I will leave all of you to think about this and argue this, it is a very interesting question, is this, so the question we are asking is, is this locally exponentially stable, great.

Now, that we have seen this we want to look at this very nice tiny little example. So, this is the system which is highly non-linear, it is, it actually looks like an oscillator, harmonic oscillator until here but then you have some non-linear terms multiplied by a constant C, some constants C. Now, we take a very very simple candidate Lyapunov functions x1 squared plus x2 squared, obviously this is positive definite radially unbounded and all the good things. And then we take the derivative as always, we are trying to do a stability analysis, so we will take the derivative.

What happens x1 x1 dot, I get x1 x1 dot and x2 x2 dot and then I simply plug in from here, it is pretty straightforward this basically it is a harmonic oscillator so these two terms obviously cancel out, they do in all of our examples if you notice and then these two start to look the same c x 1 square x 1 square plus x 2 square c x 2 square x 1 square plus x 2 square. So, actually if you combine them you get something like this c times x 1 squared plus x 2 squared whole square.

Now, if you want to conclude negative definiteness of V which is the minimum thing we would want to do. We want to conclude negative definiteness of V first of all then we well let me continue, then we would require c to be negative at least, if c is positive then there is no scope of V dot being negative definite it should be obvious to you, because this quantity is positive, in fact this quantity is also radially unbounded, this quantity is also radially unbounded.

How? You can simply see this by looking at this it is greater than equal to c x 1 4 plus x 2 4 but it is actually equal to this plus 2 x 1 square x 2 square but this is already greater than equal to 0 so I can say this is greater than equal to c x 1 4 plus x 2 4 which is of course radially unbound or the negative of it is radially unbounded.

So, depending on the sign of c and if c is positive, well, wait a second, I think if c is negative in fact this is not correct, this should be c is negative, if c is negative then V dot is actually negative definite, in fact radially unbounded negative definite and therefore the equilibrium point that is the origin is globally uniformly asymptotically stable.

But if c is positive then 0 is unstable, we will need something called Lyapunov instability theorems. We can actually show that the origin is unstable if c is positive, but we will need Lyapunov instability theorem which we did not state but that is part of your exercise, that is part of your exercise, you have to find the Lyapunov instability, an instability theorem and use it to show that the system is in fact unstable at the origin if c is positive.

Now, if c is negative one might even ask why is it not exponentially stable? If you look at this function here this is radially unbounded, this is also radially unbounded if serious c is negative, so in the negative direction, so V dot, minus V dot is radially unbounded. But if you look here which one what are the class K r functions so again phi 1 equal to phi 2 will still be half x 1 square plus x 2 square, since there is no function of time I can directly use this, this is a class K r function.

And phi 3 will unfortunately be something like minus modulus of c because c is negative x 1 4 plus x 2 to the power 4. Now, notice that this are different powers, they are similar looking polynomial terms but they have different powers. So, you can never compare them by a constant because what will happen as x1 becomes larger and x2 becomes larger this will become significantly larger than these terms and you cannot compare them by the same constant anymore.

Notice, I only get to choose a single constant to compare these class K r functions with that is I can only I am sorry I think it is here, I only get to choose this gamma 1 and gamma 2 for all 1 gamma 1 and gamma 2 for all y, it is not a function of y. But I cannot do that in this case because x 1 4 will start to dominate x 1 square significantly and similarly x 2 4. And therefore, there is no way that they are, so they are not, they are not same order. Therefore, there is no way I can get exponential stability but I do get global uniform asymptotic stability, great.

So, what have we looked at today? We sort of wanted to complete our discussion on Lyapunov stability and we have done that. So, we were left with exponential stability theorems, so we looked at local exponential stability and global exponential stability, both of which augment asymptotic stability in the sense that they give you a rate of convergence and this requires us to know about similar, same magnitude class K and class K r function, so we did define that.

We also look back at our examples to see which ones of these, which ones of the examples we had already seen were in fact exponentially stable and we found that the damped harmonic oscillator was in fact exponentially stable. We also looked at a new example and you have an exercise of course and we saw that this new example of course is not exponentially stable but can be globally uniformly asymptotically stable if you have an appropriate constant c.

Excellent, so we will continue, we will look at a slightly more involved topic here on and it basically sort of takes the knowledge of persistence of excitation to talk about new exponential stability theorems for linear systems and that is something that is rather useful in adaptive control and parameter convergence analysis, great. That is all for today. Thank you.