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Non-linear Adaptive Control
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Week 4
Lecture 23
Lyapunov Stability Theorems - Part 2

Hello everyone. Welcome to yet another session of our course on Non-linear Adaptive Control. I am Srikant Sukumar from Systems and Control, Indian Institute of Technology, Bombay. We are as always in front of our motivating image of this rover on Mars for which we hope to be able to design algorithms that can drive such systems autonomously.

So, last time we had started our discussion on the Lyapunov Stability Theorems. So, as we had stated already, these are the rather seminal results in the field of non-linear control and without the advent of these, it would definitely have been impossible to prove stability of non-linear systems such as these rovers on Mars, quadrotors, drones, electrical power grids and so many other networks of oscillators and things like that. So, these are very very critical and fundamental results. So, we saw, of course, the first two of these results. We said that we first start with what is called a candidate Lyapunov function. And what is a candidate Lyapunov function?

It is a function with two primary properties that it is a C1 function because of course we need to take first partials and we want it to be continuous. And second that it is positive definite in some domain around the origin. So, if these two properties hold true then we said that these functions are candidate Lyapunov functions.

And for candidate Lyapunov functions, if these two bullet points are considered. The first one says that if the derivative V dot which is defined using the standard directional derivative that we saw. And if this V dot is less than equal to 0 or V dot is negative semi-definite, then the origin is stable.

And on top of this V dot being negative semi-definite, if V is also decresent, this is the third function property that we had discussed, if V is also decresent on top of V dot being negative semi-definite then the system is uniformly stable.

And then we of course started to look at some examples. We first looked at the simple standard harmonic oscillator. We saw that it has phase plane portrait which is essentially just circles around the origin.

And we proved using this V equals x1 squared plus x2 squared by 2, that V dot turns out to be exactly 0 hence it is negative semi-definite. And this is essential enough to claim that the origin is in fact stable.

We also saw that this V was motivated by the phase plane portrait which is essentially circles. That is, if you start at any point, you just follow a circle around the origin starting at that point.

So, then we started to look at a slight modification of the system. In fact, a very slight modification of the system. And we started to encounter some serious issues. The first thing to observe was that we tried a couple of different candidate Lyapunov functions. So, this was the second one, in fact. This particular choice was the second candidate Lyapunov function. But we saw that with this also we were actually getting V dot is x2 squared by 2 which is positive semi-definite.

It is not positive definite, remember, because it does not contain x1 and we had very explicitly mentioned that if all the states of the system do not appear in the function then it cannot be definite. And therefore it is only positive semi-definite but still it is really bad because we only care for V dot being negative semi-definite or negative definite. So, we were not able to claim anything because this was probably not the correct Lyapunov function. We do not even know.

Now, it turns out that even for simple modifications of this harmonic oscillator, which is just division by some element function of time here as opposed to this guy, it is really really difficult to solve this system. It turns out that it is not easy to actually analytically solve the system in order to conclude stability either.

So, you can see how life can become really complicated even with simple non-linear systems and hence the rather difficult question of analyzing non-linear systems for stability. So, this is always been a rather rich area of research and continues to be simply because of this reason. But that every single non-linear system poses a new challenge as far as control design is concerned as far as stability analysis is concerned. And because of this it continues to remain a rather interesting challenge for researchers such as us.

Now, what can be said about this system? Is that if I look at this system very carefully and I look at this particular term, as time becomes really large, this denominator becomes really large.

Therefore, irrespective of what is this x1, this quantity starts to inch closer and closer to 0 for very large values of time. This quantity starts to inch closer and closer to 0. And what is this quantity? This is actually the derivative of x2.

So, if I look at the phase plane plot, I have actually made a picture here you can look at this picture, and on the x-axis is x1 on the y-axis is x2. This is how we have always done it in the phase plane portrait. The x-axis displays x1 and the y-axis is x2.

Now, what happens for very large time? Is that x2 dot becomes 0 that there is the, these are called, these lines that have drawn are essentially the velocity lines, in some sense. These are actually called the vector fields and these are called the vector field. And essentially it is plotting the right hand side of this equation. And this right hand side actually indicates how the states are going to move. So, this is the velocity line. So, if I look carefully at this.

So, in fact let me first mark this lecture. So, I know that I am actually restarting here. This I believe is lecture 4.5. So, if you look at this vector field, what we are trying to do is that we are trying to plot this vector field which indicates how the states are going to change in this phase plane portrait. So, this is, this essentially gives you the velocity lines to see how the states will move.

Now, what I have done is I have only tried to plot it for large values of time. Because what happens at large values of time? This becomes 0. So, the derivative of x2 does not is 0. So, x2 does not change. And the derivative of x1 is exactly equal to x2. So, if you look at this picture, this exactly what it is.

On this side, on on the top side, x2 is positive therefore the derivative is, the x1 velocity is positive. So, that is what it is. All the x2 and velocities are positive. And in fact as you go up up up further, these velocity lines are longer and longer. But x2 dot is 0 therefore it is actually orthogonal to x2, there is no change in this direction. There is no change in this direction.

And similarly, if I go downwards, x2 is negative therefore all the velocity lines are this way. And of course, it is like there is no change in x2 again. Now, what does this indicate? This indicates that as x2 becomes as time goes to infinity, what you have is that your states keep moving in this direction or in this direction, depending on how far you are from the origin in the vertical direction. You move accordingly faster or slower but you keep moving in these directions. So, if I start here, I will just move away from 0, again I start here I move away from 0.

So, it is clear that this system is not stable from this particular phase plane construction. But again, this is not conclusive evidence or anything. Because like I said, phase plane should not be used for concluding stability because I cannot possibly consider all the cases. However, in this case, I have sufficient evidence to see that it is not stable because I can at least see some initial conditions.

Some states which will end up at this x2 and then they keep moving towards x1 equal to infinity. So, in the x1 direction they will keep moving to infinity in the positive or the negative direction. Therefore, this system is not stable. Because whenever your system states just start to go to infinity in any direction, in this case in the x1 direction, maybe in the x2 direction they remain bounded, but in the x1 direction they move to infinity, so the system cannot be stable.

Because if I give you any epsilon, you need to be able to find a delta such that you remain within the epsilon ball. In this case, if I draw any epsilon ball like this say I will not remain within it. Because my states are trying to go outwards like this. And this is true for large time. For small time something else happens we do not even care what happens for small time because I can always keep increasing time and see that this does not hold true anymore at large time. So, the system is not stable. However, I could not find any conclusive evidence to the Lyapunov construction and it is not very easy to solve this analytically.

So, even for a really really tiny modification, this is not even a non-linear system this is in fact a linear time varying system. We were really stuck. So, this should help you understand the enormity of the stability question. So, next we sort of try to, we constructed a sort of concocted modified version of this and that is this. We have constructed a modified version of this. And what is this modified version?

In this case, you have x1 dot is x2, x2 dot is the same minus x1 over 1 plus t. But I have now added minus x2 over twice 1 plus t. This is sort of a trick if you may. And I choose the same candidate Lyapunov function as here. It is the same function. What happens?

I take the derivative, it is x1 x1 dot plus 1 plus t times x2 x2 dot plus x2 squared by 2, where I have taken the derivative of this with respect to time.

Now, if I substitute for the derivatives from my dynamics, this is x1, x2 plus 1 plus t times x2 x2 dot. So, this is minus 1 plus t. Now, so, this is a minus x1 minus x2 over 2 plus x2 squared by 2. And what happens? You can see that these two cancel out. So, I am left with this is x1 x2 minus x1 x2 from here. And finally, minus x2 squared by 2 from here.

And then plus x2 squared by 2 which is exactly equal to 0. So, I have V dot is less than equal to 0. So, from this I definitely have stability xe equal to 0 is stable, because I took a candidate Lyapunov function. How is this a candidate Lyapunov function? Notice that V is positive definite, it is not difficult to see that V is positive definite. Why? Why? Because if I take t greater than equal to 0. So, for all t greater than equal to 0, of course. So, for all t greater than equal to 0, what do I have?

If I plug in any non-zero x1, x2, non-zero state. So, if if x is non-zero at least x1 or x2 is non-zero therefore V is always positive. This is a positive definite function. So, therefore V was positive definite, it is C1 continuous and V dot is negative semi-definite. So, I have satisfied all the requirements for stability. So, the now the question is, is the system uniformly stable at the origin?

So, we notice that V is not decresent. Why? Notice that this is a continuously increasing function of time. So, if I claim, I cannot ever claim that V is less than equal to phi norm x because where phi is a class K function. Why? Because, phi is just a function of the states. So, if you give me any such phi, what I will do is for that fixed phi, I will fix a state x.

Some very small value of the state it does not matter I fix any particular value of the state. It does not matter. I fix any particular value of the state. So, the right hand side becomes a constant. Once I fix x the right hand side is a constant. But notice the left hand side is an increasing function of time. x1, x2 are constant, no problem, just as the right hand side.

But this is an increasing function of that. So, I will keep pushing up time. I will keep pushing up time. I will keep pushing up time. So, that the left hand side can never be less than the right hand side. Because the right hand side was some constant it does not matter, how large it is. I can always push the t larger and larger to achieve V greater than phi naught x.

So, therefore V is not decresent and therefore implies xe equal to 0 not uniformly stable. It is not uniformly stable. This is a rather interesting result. So, we have a system here which is being proven to be stable. It is actually a modification of the harmonic oscillator.

But not just one term, one time modification but I also added some additional state term. So, it turns out that this is stable but not uniformly stable. At least I cannot claim uniform stability with this particular Lyapunov function. So, let us be careful here. We cannot prove uniform stability with this particular Lyapunov function. It may be possible to prove it with something else. But in general, it is not too difficult to see that the stability will not be uniform.

Because what happens is that this term that is all these nice term, for example this term. See if you notice this system was unstable. So, the stability is obtained due to this additional term. This is like a damping term here. Because until this term was missing, the system was unstable. So, this damping term sort of helps us get stability in this case.

Now, this damping term becomes smaller and smaller with time increase. So, therefore the stability property is sort of time dependent. It will not be possible in general to prove uniform stability in this case. So, anyway, so that is an aside but the point is with our Lyapunov theorems, we cannot prove uniform stability. We can only prove stability for this particular system. Excellent. So, I think we have some idea of stability and uniform stability and we have seen some examples.

So, now, we can move forward to the next sort of results. So, in fact, I should actually write here. So, here lecture 4.5. So, the next result talks about local asymptotic stability or just asymptotic stability. So, the next two results are local asymptotic and local uniform asymptotic. So, what do you require?

Earlier, we had only talked about semi definiteness of V dot, here, we make it more stringent. So, we have stronger and stronger properties as we go downwards in these bullet points. So, in this case, we have, we require that V dot is negative definite. So, if V dot is negative definite for a candidate Lyapunov function, then the origin is locally asymptotically stable.

Similarly, if I add the decrescence property to the negative definiteness then I get uniformity. So, if V dot is negative definite and V is decrescent then the origin is locally uniformly asymptotically stable. So, we also use acronyms very commonly to denote these properties because these get rather long set of words. So, we do not always want to write them. So, we calls uniform stability US, asymptotically, asymptotic stability AS, uniform asymptotic stability UAS.

So, we have specialized from stability to asymptotic stability by adding negative definiteness in place of negative semi-definiteness. And similarly, from uniform stability to uniform asymptotic stability. And just this negative semi-definiteness gets replaced by negative definiteness. So, this is the difference.

So, let us see again some examples. Let us see some example. So, if I take again this system, let me try to construct a nice example which will of course help us prove some nice properties. Let us see. Let me. So, suppose I have a system which looks like this and I want to do something, I want to take an interesting Lyapunov function, V x1 x2 is equal to say half x2 plus x1 squared.

Now, let me see this. So, V x1 x2 is half x1 plus x2 squared plus half x2 squared. No, think should be x1 square. So, then I take the derivative of this. So, it should be evident that this is positive definite. Why is it positive definite? So, question is why is it positive definite. So, if I want to make this, let us look at where this can be 0. This can be exactly 0, if and only if x1 exactly 0 and x1 plus x2 exactly 0, which means x2 exactly 0.

So, this origin is the only place where this function can be exactly 0, everywhere else it is strictly positive. So, therefore this is a positive definite function as per our definitions. So, if I compute V dot I will get x1 plus x2 times x1 dot plus x2 dot and plus x1 x1 dot. So, this becomes x1 plus x2, let me see if this works. And this x1 dot is x2 and x2 dot is minus x1 minus x2 plus x1 x2. So, let us see. So, this is equal to x1 plus x2 times minus x1.

In fact, let me sort of keep it as is, x2 minus x1 minus x2 plus x1 times x1 plus x2 minus x1 squared. So, I have basically written this guy as this, just by writing x2 as x1 plus x2 minus x1. And then I will club this guy to this. So, this will give me x1 plus x2 times x1 plus x2 minus x1 minus x2. So, let us see, I want to make this slightly different.

So, what I will do is, let us see, I will make this I will make my life a little bit easier. Let me say this is, I mean I can always change the candidate function here, that will give me. What will that give me? I think I should have taken something like half x1 or something. So, well that would have helped.

So, what I will do is make a slight modification here. I will make a slight modification here. I will add a k, here. And I will see where all the ks propagate. k is here, k is here, similarly, k is here, k is here again, the k continues here, and the k continues here. Just I am adding this k just, so, I can have a little bit more of a control on what shows up here.

So, in this case, what I will do is instead of I will take a k here and I will subtract the k here. So, and then I will have a k again here. So, this becomes x1 plus kx2 minus x1 minus x2. So, this is still, let us see this is still a bit of a problem. Because I still want a negative in the x1. I would still like a negative in the x1. So, I am just trying to manipulate some of this numbers.

So, that we can get an appropriate. So, that we can get an appropriate sort of equality here. So, that is the idea here. So, suppose I make it, I apologize. So, this is say, 1 by 4. So, this becomes 1 by 2. This becomes 1 by 2. This becomes 1 by 2 and this becomes divided by 2. So, this also becomes divided by 2. So, what I get here is k x1 plus x2 times minus x1 by 2 minus 1 minus k x2. And this is and I want this to be actually equal to k x1 plus x2.

So, let me sort of write this as 1 minus k k x1 plus x2 minus this is x1 over twice 1 minus k twice 1 minus k plus x2 and all I now need is that k be less than 1. So, that this is positive. And I need this quantity to be equal to k. So, I want 1 over 2 1 minus k is equal to k. So, if I satisfy these two, if I can satisfy these two, this is simply equal to 2k minus k square equal to 1 implies k squared minus k plus 1 equal to 0.

So, if can I actually satisfy these two it is the question. So, this is, but this is not, I am not sure this is possible because this is going to give me k equal to 1. So, this is not good. So, this is not good yet. So, anyway what I will do is I will, so this construction is right in spirit, you can see that this construction is right in spirit, the only thing is I have to be careful about choosing these constants.

So, what we will do is we will continue next time and we will actually choose these constants appropriately. So, that we can claim asymptotic stability. So, anyway. So, what did we do today was to sort of look at the next two definitions which is asymptotic stability and uniform asymptotic stability. And we are in the process of working out the example to prove asymptotic stability using a Lyapunov construction. This is what we will continue next time. Thank you very much.