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Nonlinear Adaptive Control

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Week -04
Lecture 22

Lyapunov Stability Theorems - Part 1

Hello everyone, welcome to yet another session for NPTEL on nonlinear and adaptive control. I am Srikant Sukumar from systems and control IIT Bombay, we are as always in front of our very nice representative image of rover on Mars, which is driven by algorithms that are allowed for autonomous motion of the device on extra terrestrial surfaces, and we hope to be able to learn to analyse and design algorithms that drive systems such as these towards autonomous operation.

So, what we have been doing until now is look at many different function properties like we started with you know, first looking at a few function classes, at the end of last week, and then based on these function classes, we defined a few different notions The first was the notion of definiteness, which is where, which is sort of important for talking about asymptotic stability, then, we sort of looked at the notion of radial unboundedness, which encapsulates or which sort of is going to help us conclude global stability.

And finally, last in the last lecture, we looked at the properties of decrescence. In each of these lectures, of course, we also looked at examples, we also sort of looked at a few relatively easier conditions to verify the corresponding definitions. So, in a few cases the definition themselves are not too easy to verify, and therefore, we used some easier conditions to verify these.

So, once we have these properties, and of course, in the end, we also saw the most basic property which is semi-definiteness. So, once we have these properties sort of well understood, we are ready to state the most seminal results in nonlinear control theory and these are the Lyapunov theorems. So, what the Lyapunov stability theorems.

So, that is where we want to begin our discussion today. So, we are in the 4th lecture of week 4. So, before we state the theorems themselves, it is important that we look at a few, you know, what is the setup for the theorem and things like that. So, we assume that we are talking about a non-linear system of the form x dot is f t, x. Of course, the time argument is being made explicitly explicit here.

And the function f which is the vector field is allowed to depend explicitly on time and also on the state. Further we of course assume you know, reasonable domain for the function f, which is some initial time to infinite time some ball around the origin and it maps to some Rn. Because x dot has to be in Rn and we of course, have some initial condition xt0 is x0.

Now, without being loss of generality, like we have done, so many times before, we are also going to assume that xc equal to 0 is in fact an isolated equilibrium. So, in fact, we will say isolated will make that formal that this is in fact an isolated equilibrium point. We already discussed why we will have some issues if the equilibrium is not isolated.

So, we first assume that 0 is an isolated equilibrium point and that is the equilibrium point which we are sort of interested in analysing f is assumed to be locally Lipschitz continuous in this ball, which therefore, exists ensures the existence of a unique solution. Because, without existence of a solution there is no point in even talking about stability. If there is no solutions that exist or no unique solutions that exist then talking about stability becomes a serious challenge.

And further we define V dot using what is called the lie derivative or the directional derivative. What is the idea? If you look at all these functions V that we defined they are functions of time and some variable x. So, when I take the derivative of that, I get partial of V with respect to t and partial of V with respect to x times an x dot.

Now, if I substitute for x dot from this particular vector field, then it becomes what is called a directional derivative. So, V for example, I can use the same construction of V to analyse many non-linear dynamical systems. However, the derivative of V will depend on which dynamical system I was choosing to analyse. And therefore, V itself is not particularly connected to any dynamical system.

So, this we should keep in mind the V construction itself is just a function of t and X, it is not necessarily connected to a particular dynamical system, it is when we take the derivative in the way we have defined here, which is also known as the directional derivative is when we introduce the dynamics in the form of the expression for x dot right here and that is why it is called directional derivative that is derivative of V along 10.1 is also how we sometimes say so, this is sometimes also denoted as derivative of V along our trajectories infact of 10.1.

So, this is very important, it is very key that we remember this setup that the Lyapunov about the function V that we have looked at until now to evaluate definiteness and to evaluate radial and boundedness, et cetera. It is just merely a function of time and state x. We do not attribute any dynamics to it.

When we take the derivative as per this definition, here, that is when the dynamics actually filters into the equations and therefore, it is called the directional derivative. So, with this setup, we are in fact ready to state the Lyapunov theorems. You will see the statements themselves are very easy. So, we will of course, state them and of course, start to look at examples parallely.

So, these are called Lyapunov stability theorems, and whatever results we get using these Lyapunov stability theorems are called Lyapunov's direct method. So, we suppose for each one of this that there exists a function V t, x which is mapping t0 comma infinity cross Br to R those are same very similar earlier we were using R plus here, but here we are specializing it to say that we only care about the time starting the initial time of the dynamics.

So, therefore, we have a t0 here, it does not really bother us what happens before t0. In most cases, things will be the function itself will be nice enough before t0, but since we do not care, since our dynamical system is going to be initialized that a particular t0. So, we take the domain and time as t0 comma infinity and this is good enough for us.

The states of course, are restricted to some ball around the origin. Again, this is for the local results. And now, notice that the solutions are also assumed to exist in this ball only for the dynamical system. So, we have consistently kept the same size of the ball around the origin in which there is the solutions also exist and in which the function V is also defined.

And we assume that it is positive definite. You see this notation and we have already introduced this notation, this notation implies that V is positive definite, and what do we know? We know that these two conditions together make it a candidate Lyapunov function, A function which satisfies these two conditions. That is well these two conditions in the sense let us see, in fact, this is not complete, it should be more I apologize this should be and there exists a C1 function V. So, V has to be C1 at least because we are going to take derivatives of it and partials of it.

It has to be once continuously differentiable at least. So, any function which has these two properties that is it is a C1 continuous function and it is V is positive definite, then it is called a candidate Lyapunov function and using a candidate Lyapunov function, only, we can state the Lyapunov theorems.

So, once we have this candidate Lyapunov function, then once we if we evaluate the derivative, and it turns out to be negative semi definite then the equilibrium is said to be stable. It is not said to be stable and the equilibrium is stable it can be proven and the second statement, we are just going to focus on these two statements, very carefully. The first statement says that if V dot is negative semi definite that is V dot is less than equal to 0 then the equilibrium is stable. The second statement says that in addition, if V dot negative semi definite and V is decrescent, then the equilibrium 0 is uniformly stable.

So, V dot is negative semi definite only than 0 is stable, if V dot is semi definite and on top of that, you have V to decrescent and then the equilibrium is said to be uniformly stable. So, take a candidate Lyapunov function which means that C1 function which is also positive definite, and if it is negative semi definite, we have stability, if it is negative semi definite and V itself is decrescent then we have uniform stability. So, remember V when we talked about this, all these conditions, positive definiteness and so on and so forth and negatives semi-definiteness, negative definiteness decrescence, we had mentioned that the decrescent is associated with uniformity. As soon as I add it decrescence I got uniformity.

So, we want to of course, start looking at examples right away, we want to look at examples right away. So, the first one that is already written out here you can see is x1 dot is x2, x2 dot is minus x1. So, do you know what this system is? This is an simple what is called simple harmonic oscillator.

What does it look like in the phase plane, if I make a phase plane plot of the trajectories of this system, I if can make a phase plane plot of the trajectories, then what does it look like make it bigger, easier to draw. Looks like a bunch of circles. So, the state space portrait of this system looks like a bunch of concentric circles.

So, that is if I start my trajectory anywhere, if I start my trajectory anywhere. I will just continue on this circles forever. I will never leave the circle. So, this is what the state space trajectories look like. So, look like a nice enough benign system. So we of course want to look at the stability analysis. So, what do I do? I pick my V as half x1 squared plus half x2 squared. So, what do I know about this? I know that this is V is positive definite, V is a C1 function and it is once continuously differentiable at least. In fact, it is a smooth function.

So, I can differentiate it as many times as I want its polynomials. So, it is already a candidate Lyapunov function because it is C1 and it is positive definite. So, excellent. So, I am ready to apply my Lyapunov theorem. So, what do I want to do now? I want to compute V dot. So, what is V dot? It is, so V is not a function of time at all. So, it is only a function of state. So, all I have to do is del V del x times x dot, which is, in this case, x2 and minus x1.

So, evaluating this is pretty easy. In fact, I mean, I do not do it with this formula. And I do not recommend you do it with the formula either. Because if I just take derivative using the standard product rule, it just becomes x1 x1 dot plus x2 x2 dot. And what will I do? I will just substitute for x1 dot from here, and x2 dot from here and that is the directional derivative.

Because all I am doing is substituting from the current dynamics. And this is not difficult to see that you will get x1 times x2 plus x2 times minus x1, which is exactly 0. And therefore, it is also less than or equal to 0 and negative semi definite. So, we had a candidate Lyapunov function, which is x1 squared plus x2 squared over 2. And we have shown that V dot is less than equal to 0. So, what have we shown according to the Lyapunov theorem we have shown that x1, x2 equal to 0, 0 is stable. Now, let us look at geometrically what is happening a little bit for this system.

So we have already applied the Lyapunov's theorem, great, one Lyapunov theorem we have already been able to apply. So, we have already learned to some extent, how do we apply it, we start by choosing a V. And we have to of course, make sure that it is C1 at least, and it is positive definite, only then it is a candidate Lyapunov function on which Lyapunov theorem can be applied.

And then I compute V dot, instead of doing this sort of vector calculation, I simply take a product rule, I mean, just take derivatives directly here, x1 x1 dot plus x2 x2 dot, and I substitute for the derivatives from the dynamics because I am taking the directional derivative along this dynamics. And this is 0. So, you can see it is super easy. It is turned out to be very very easy, within like, a couple of simple steps, I have concluded stability in the sense of Lyapunov.

Now, one thing that should be obvious to you is that this is a very simple system. Because this is simple harmonic oscillator. It is a linear system. So, I can in fact, solve for the dynamics, and what will be the solution? It will be just sinusoid and sines and cosines is what will be the solution, that should be obvious. If you are not convinced, I would recommend that you solve this.

So sines and cosines are what will be the solution, but I can promise you even finding and writing out the solution for this will take more number of steps than what you just did. So, this was significantly simpler. And what else? We reduced the analysis to analysing a scalar function. So, geometrically, what is the relevance of this x1 squared plus x2 squared by 2? If you look at the phase plane portrait that we do, again, I could not have drawn the phase plane ported unless I could actually solve the system. I can do it numerically too.

But of course, you then you I mean, if you want to actually conclude stability from a phase plane portrait, you will have to draw the phase plane portrait corresponding to every initial condition, which is usually not possible, which is why we rely on Lyapunov analysis. But in this case, the phase plane portrait does give us some insight like, what is this insight? If you look at this, what are these circles these are concentric circles. So my system is always evolving on a circle. And what is the equation of a circle? It is x1 squared plus x2 squared equal to C, there is a circle centred at the origin.

And these circles are centred at the origin. How that does not seem like it unfortunately, from this picture, but they are centred at the origin. So, the solutions follow the equation x1 squared plus x2 squared equal to C equal to a constant. So, if I start on a circle, start at any point, just start following a circle. Now, if you look at my Lyapunov function, or candidate Lyapunov function that I took, this is simply the same quantity divided by 2. This guy is just this divided by 2.

So, just a scaled version of this and therefore, it makes sense that when I took the derivative along this system, it came up out be 0, because all my trajectories always lie on a circle. And therefore, x1 squared plus x2 squared at each point in time is always a same constant. And therefore if I take the derivative here, if I do d dt here then I do d dt here, and this is just 0.

Therefore, V dot coming out to be 0 is no magic. Coming out to be 0 is not really magical or anything. So, this is some actually somehow the energy of the system that I have encapsulated in this V and so therefore, this is coming out to be a stable system. Excellent. So, I can conclude the same thing again, using, like I said, solving this system, this is significantly simpler.

Let us, go forward and look at, you know, another example. This is just a slightly twisted version of this where I have included some time dependence. But I have included some time dependence. Now, the question is, can we actually find an appropriate Lyapunov function is the sort of the first question, sort of the first question, may or may not be easy, even I do not know if we can do that. So, let me see. So, this is one of the sort of issues that everybody deals with is that an appropriate choice of Lyapunov candidate Lyapunov function must be chosen.

And whether this is possible always is not very clear. Unfortunately, it is not too clear. So, that is, what it is. So, let me see if I can actually choose and appropriate Lyapunov function. In this case, it is obvious that if I choose V as my earlier one, if I choose V as x1 squared by 2 plus x2 squared by 2, I have a problem because this is x 1 x1 dot plus x2 x2 dot, and this is x1 x2 minus x2 x1 divided by t plus 1. So, these do not cancel out any more. And that sort of results in a problem.

So, let us see if we can do something else. In fact, maybe I will not be able to do something for this case. I am going to slightly change this example to this, suppose I change this example to this in this case anyway nothing worked out. So, suppose I will erase this guy. So, this is not a good enough choice. So, let me see it, is this something for which I can find a Lyapunov function. See, I am already starting to struggle a lot, you see.

So, I do not think even this will help me because suppose now I do something naive, like, I choose 1 plus t x1 squared by 2 plus x2 squared by 2. Then what happens is I get x1 x1 dot plus x1 squared by 2 plus x2 x2 dot. I got this second term here, this is I am sorry wait a second, I think I miss something here.

I think I miss something here, this should be 1 plus t still has to be multiplied here. So, the first term here is obtained by taking the derivative of this x1 square. The second term here is obtained by taking the derivative of this guy. And third term is of course, the derivative of this. Now, if I substitute, right now if I substitute for my dynamics, I will get x1 x2, I will get 1 plus t x1 x2 plus x1 squared by 2 minus x1 x2 divided by 1 plus t.

So, I do not think this worked at all, I Apologise, this is wrong. This is not what I wanted. I wanted it to be I apologise I wanted it to be half x1 squared plus the other way around, actually, 1 plus t x2 squared I am sorry, I wrote it the other way around. And if I tried this, this is x1 x1 dot plus 1 plus t x2 x2 dot plus x2 squared divided by 2.

And now if I substitute here, I get x1 x2 from the first term from this guy, then I get plus 1 plus t x2 times minus x1 over 1 plus t plus x2 squared over 2. So, something nice does happen, this 1 plus t cancels out here and therefore this guy cancels it, this guy. This is why I chose something like this. But then something really bad also happens because I end up with x2 squared by 2. I end up with x2 squared by 2. This is not nice. To say the least. This is not nice to say the least.

So, this is where things can get wiry. It is not that easy. Now, the question is this actually a stable system? And is this an actually a stable system or not? In this case, finding the solution is also not going to be very easy. So, now see, what has happened is that this is not, is in fact it turns out that I am sorry, it was not careful here. I should say implies V dot is equal to in fact, this turns the tree positive semi definite. And this is not good at all. Because if you notice all that Lyapunov theorems rely on V dot being negative semi definite or negative definite. We do not want a V dot to be positive definite or positive semi definite at all.

So, what is the outcome? The outcome is we do not know cannot conclude on stability yet. This is one of the issues with the Lyapunov functions. So, we should, you should be aware of it from the get go. Just because I choose a function V in fact was second choice just because I chose the function V and I found out V dot, which did not satisfy the nice properties that I want it does not imply that the system is not stable, or asymptotically stable.

It may just be the case that I was not good enough at selecting the Lyapunov function. So, the Lyapunov function is still a hunt you have to be able to find the appropriate Lyapunov function. You have to be able to find the appropriate Lyapunov function and that is still a hunt. And that is why I sometimes say it is actually Lyapunov part. There is an arch to finding the light Lyapunov function, anyway.

So, what we looked at today was the beginning of our Lyapunov theorems. We saw the theorem on stability and uniform stability, very, very, very simple. That is what we have noticed. And we have tried to work out some examples, we in fact failed in finding an appropriate candidate Lyapunov function for a very seemingly simple looking system.

And so we will continue in this way we try to construct Lyapunov functions, we try to conclude something about the stability for systems such as this. So, that is going to be the plan subsequently also that we going to look at a few more examples and also continue the current example. Alright, that is it folks. See you again. Thanks.