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**Nonlinear Adaptive Control**
**Professor Srikant Sukumar**
**Systems and Control**
**Indian Institute of Technology Bombay**
**Week -04**
**Lecture 21**
**Decrescent Function**

Hello everyone, welcome to yet another session of our NPTEL on nonlinear and adaptive control. I am Srikant Sukumar from systems and control, IIT Bombay. We are as always in front of our nice motivational image of this autonomously operating rover on Mars and we the sort of advancing towards analysing and developing algorithms that we hope will drive systems such as this.

So, last time we were looking at the notion of radial unboundedness. We had started with these special functions, which we are going to use subsequently in Lyapunov theorems. And the first property that we looked at was the property of definiteness. So, we looked at positive definite functions. And then last time, we were looking at radial unbounded functions in some detail.

So, as always, we sort of gave some definition for radial unboundedness which is sort of an extension of positive definite functions. And we just like in the case of positive definite functions, we also proposed some easier tests for verifying radial unboundedness. So, just like the case of positive definite functions and the definition itself may not be amenable for use in a lot of cases.

And so, we sort of use some came up with some easy tests, because the definitions themselves require us to come up with some class K and some class KR which may not always be easy to find. So, of course, also for the case of radial unboundedness we came up with this easier conditions and we saw some pretty hopefully some pretty good representative examples.

Of course, the exercises that we are providing have more of these examples and more of the examples that you have to consider and discuss definiteness and unboundedness et cetera for. So, this was the notion of radial unboundedness. So, we also a pointed out that positive definiteness was connected to local stability notions and radial unboundedness is connected to global stability notions, this is again something that we will see subsequently.

Now. There is also you know, if you go back and sort of remember the definitions that we had looked at, we also looked at the notion of uniform stability. And this uniform stability notion are what are connected to decrescence. So, that is what we are going to look at today. So, this is where we start, we say lecture 4.3, this third lecture of the 4th week.

And we want to define what is a decrescent function. So, if we as always, we have a scalar valued function, which is going to take as its arguments time. And again, just like we mentioned last time, there is really no harm in fact, it might even be better to assume that this function takes non negative real time, because that is really, that is what we are going to consider almost everywhere in this course, we have most courses in nonlinear control.

So, this is a function which takes in the first argument time R plus, and the second argument as state, which is now in some bounded ball around the origin. So, this is Br, Br we have already introduced this notation. And as usual, it maps to reals. So, all these functions are, have a similar range and domain.

And of course, we require that V t0 be equal to 0 for all t in R plus, this is again a same condition as before, remember that we really care about this function taking 0 values at the origin. Because origin is where we want to convert and this has the notion of some kind of an energy function and we want this energy to be to take its minimum value. When you are at this equilibrium, which is the origin.

So, for decrecent, this second condition that we require is the existence of a class K function such that absolute value of V t, x is less than equal to find phi norm x, for all t in R plus, and for all x in Br. So, this is, you know, sort of completely opposite to what we saw for positive definiteness in some sense. So, at least looks like it is completely opposite to what we saw for positive definiteness.

So what is the idea? It has to be dominated by a class K function. So, if you look at the picture, this should help us identify the difference. Here we have x and here, of course, we have all the different functions of x, so that you will sort of mark later on. So this is, say my class K function. I mean, again, this is only until some r beyond that, we do not care. I mean, so this is just some class K function.

And what we want is that, in fact, let me draw it in a different way. So, it will indicate that I can, in fact, go below it. No, Let us take that V. So we want this V absolute value of V to be less. So, of course what I will do is I will draw something on the other end to something like this. And we want so this blue thing is, of course, the phi norm x, and this is minus phi norm x. And we want the V to be bounded inside this.

So, we want V is allowed to go in the negative direction. But it has to remain within this sort of envelope, if you may. So, this is what does it decrescent function and this is an example of decrescent function. So, this is an example of a decrescent function.. So, as I mentioned before, the decrescent is related to the notion of uniform stability. And so basically, it somehow, if you look at this, I mean, it is not evident here. But anyway, I mean, it essentially, that is somewhere of independence with respect to time that you achieve using the notion of decrescent. So, what are the examples?

So let us look at examples and counterexamples, just like we have been doing. So, let us see, let me try to expand this and say look at some example 1. What is it? Suppose I take something like V t, x is equal to what we took for the positive definite case, let us try to see that. So, this was t plus 1 over 2 times norm of x squared. I mean you can also take norm of x square divided by 1 plus norm of x square, so that is fine.

So, we know that this is positive definite. So, V positive definite, we have already sort of discussed this a couple of times before. And because this is of course, greater than equal to this guy for all non-negative time. So it is positive definite, but is it a decrescent? No, it is not decrescent. Because there is no way I can you know, so this is sort of not very easy to argue again, because we are not giving any easy test in this case. So, it is not very easy to argue.

So, we want to claim is ask if V is less than equal to I mean, this is already non negative. So I am not using absolute value of u. So, is V less than equal to sum phi norm x. Let us, think of a simple situation. Let us say, is this true? Is this true for some positive gamma? What do you folks think? So, I have essentially taken this guy out. I am saying, does this inequality hold true for some gamma?

Because if it does, obviously, this right hand side is a class K function. And so we are done. I mean, find the right hand side is not exactly a class K function, but the right hand side is a positive definite function. So, we are fine. Where sort of you know, it is not a big deal. Because I can always say that this is, let us see less than I apologize I can always sort of say that this is less than equal to gamma norm x square and then I am done.

Because gamma norm x square is of course a class K function. In fact, it is a class KR function, so that is okay, class KR function is also a class K function. But does this even hold true? Does this hold true? So, the point is, this guy, I am going to put as a nice square around it. So, this is not let us say sorry not true. Why? Why is this not true? If you give me any gamma, for example, say gamma is 100. If you give me a really large gamma, say, gamma I equal to 100 I choose some arbitrary large gamma. What do I know? I know that at t greater than 99, V t, x it becomes greater than gamma norm x squared divided by 1 plus norm x squared.

And I immediately get a contradiction. So easy, I got a contradiction, see it very very easily. And say if you give me any choice of gamma, arbitrarily large, it does not matter, I can always find a t such that it will start to dominate this function rather than being less than this function, it in fact, starts to dominate this function. And that is a problem. And so you can see that this V is not decrescent. So, this is this is V not decrescent. So, this is positive definite. But it is not that decrescent.

So, let us look at another example. And say the other way around. Suppose I take V t, x I am going to make it bigger. Suppose I take my second example with V t, x as equal to I sort of divide. So suppose I take 1 over 2 t plus 1 norm x squared divided by 1 plus norm x squared. In fact, I will keep my life simple and I will just get rid of this guy.

I will just say it is norm x square. Because anyway, it does not matter whether it is positive definite or radially unbounded. Because we are more concerned with decrescence here. What can I say? I know that this is for sure less than norm x squared over 2 for all t greater than for all t in R plus. Why?

Because as t becomes larger than 0, this is definitely smaller than norm x squared by 2.That is obvious. So, implies decrescence sorry but V not positive definite. Why? Let us, look at the easier condition. What do we require? We require V to be strictly positive for all norm x not equal to 0. On the face of it, it looks like. But if I take any nonzero x, the problem is I have division by t plus 1 this guy.

So, what happens? Again I will say no so limit as t goes to infinity, 1 over 2 t plus 1, norm x squared is 0. It does not matter what norm x is, it is irrelevant, it is irrelevant what norm x is. As I pushed t larger and larger, this is going to go to 0. Therefore, V is not positive definite. So, we have some sort of contradictory examples.

We found that for when we you know, sort of I give you an example, where if it was a function of state and time, where V was the function of state and time and it was decrescent like this case here. It turned out that it was not positive definite. Because if it was upper bounded by a class K function, I could not lower bound it by a class K function.

I mean, I did not really see it in terms of lower bound in class K, but I did the equivalent easier test. On the other hand, if whenever I could lower bound it by a class K function like this, this is, you know, whenever I could prove that it was positive definite, which we did already in the previous lectures.

We could show that it was not decrescent. Of course, in this case, we only showed with a particular example of a class K function, we could have said that it is dominated its upper bound by some really large class K function. But the point is, whatever function of x I choose here, remember that the right hand side has to be independent of time.

So, whatever function of states I choose, this has to be true for all x, notice. So, if I choose really small x, then this is right hand side is potentially not very big. But as I keep pushing my time, up, and up and up, V t, x is always going to dominate something like this. Any function of only the state on the right hand side.

So although I chose this particular example, to verify that it is not decrescent, and it does not matter, I could have chosen any function of state here. And this will always dominate this guy, as t gets pushed up as t sorry as t goes to infinity. So, is was not decrescent. So, the example where I had positive definiteness, I did not get decrescence. The example where I got decrescence, I did not get positive definiteness. So, it is makes the question, is there an example of a does that even exist an example of a V, which is function of both t and x, which is both decrescent and positive definiteness.

So the answer is yes, answer is yes, it is not that difficult. And what is it? You would of course, first choose the nice positive definite sort of structure. Let us see, I will choose something like this. I will keep a norm x squared over 2 here. And what I will do is I will take something like a bounded function of time, if you may I will take a bounded function of time. So, this is something like 1 plus sine squared t.

So, what do I know about this V t, x? I know that V t, x is definitely greater than equal to norm x squared by 2 implies V positive definite because I could find lower bounding class K function. Why this is true? Is because, the lowest value this can take a 0. Largest value is of course 1, but the lowest value is 0 and so it dominates this class K function.

Simple, so this becomes 1. So now I also know that this has the largest value, largest value this can take is 2 because sine squared t is upper bounded at 1. So, this whole thing can at most become 2. So, I also know that V t, x is upper bounded by norm x squared implies V is also decrescent upper bounded by x squared, why? Because the largest value this can take is 2. So, the 2 and 2 we are cancel. And that is the largest possible value of this function.

So, I have bounds on both sides. So, there do exist, of course, functions of both state and time, which are in fact both decrescent and positive definiteness positive definite. So, it is not like you know, one excludes the other or something like that, let us not. I hope we are not you know, sort of getting into this preconceived notion that one does not mean the other because if that was the case, then it would never be possible to get uniform stability and asymptotic stability together, as we will see further.

So now, the other thing to remember of course, is that if you have a function, which is actually just a function of the state. So, this is obviously positive definite. So, in such cases only, so and in fact, make a remark also remark if V only dependent on x, then decrescence is free trivially decrescent. Because, of course, It is upper bounded by norm x squared this is upper bounded by norm x squared, and I am done. Hence is trivially decrescent because there is no time appearing which will not allow this domination. I hope that is clear.

And the final property which is the actually rather week property is that of positive semi definiteness. So, the positive semi definiteness is a rather simple property, it just says that, you know, anything, if as usually you have function going from time to the state. And, of course, we want V t, 0 to be 0. Further, if the function is just non negative, just takes non negative values. That is all we need for semi definiteness, if a function V takes non negative values for all values of time and state in ball then V is said to be positive definite semi definite and it is denoted by a greater than equal to 0. So, positive definiteness was denoted greater than 0. So positive semi definite is denoted greater than equal to 0.

So, these are rather simple function is basically any function taking a non-negative value. So, it means so many examples, so V t, x equals t x1 square plus x2 square is final is greater than equal to 0, V t, x equal to x1 square by 2 plus x1 squared by 4, which was not positive definite remember, is in fact positive semi definite.

Because in a whatever happens, this is never going to take a negative value. Similarly, if I choose V, all the examples that did not satisfy other properties, this is again, positive semi definite. Because whatever happens, this function is never going to take a negative value. And remember that positive semi definite semi definiteness also plays a important role in the Lyapunov stability theorems.

So it is not a class that is to be taken lightly. But of course, it is a rather weak requirement. It just requires that the function itself never take a non-negative value, that it that is all we require in positive semi definite. I hope that all made sense. So, beyond this, we are of course, ready to look at the Lyapunov theorems, I look Lyapunov theorem but we will stop here and sort of try to conclude a discussion for this session.

So, what is it that we saw? We went on to sort of look at the final property if you make and this is the property of decrescence. So, we have seen three properties or functions I mean just positive definiteness connected to a asymptotic stability, radial unboundedness, which we said is connected to global asymptotic stability, and decrescence which we have mentioned is connect to uniform stability. So, you have seen all the three properties.

In this week's lectures, we have also seen a good number of examples to hopefully help us distinguish between these properties. So, I really hope that it helps clarifies what kinds of functions are positive definite and which aren't positive definite. So, it is more important to sort of remember the ones that are not positive definite. So, we do not make any errors in our Lyapunov theorem applications

And of course, these are other nice potential functions, and we will see subsequently the setup for the Lyapunov theorem. And we will also see how these three properties of functions that we have defined, we have looked at easier tests to actually talk about these to actually verify these definitions. So, you will actually see how these three kinds of classes of functions play pivotal role in helping us state the Lyapunov theorem. So, this will be rather critical in what is going to come in next.

So, I mean, what is upcoming in the lectures now is what is the most most critical aspect of a nonlinear control, which is the Lyapunov stability theorems and which is what helps us to conclude that a system or an autonomous systems, which is what we see in the background, it performs satisfactorily in any given environment that it follows a given trajectory or reaches a particular point or has a particular orientation.

So, all of these are posted stability questions by control engineers, and nonlinear control theorists verify these using the Lyapunov stability theorem. So, what we will see subsequently are probably one of the most seminal results in nonlinear systems analysis. Excellent. So, that is where we will stop today and I hope to see you again soon next time. Thank you.