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Nonlinear Adaptive Control
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Week 4
Lecture 20
Radially Unbounded Functions

Hello, welcome to yet another session of our NPTEL on nonlinear and adaptive control. I am Srikant Sukumar from systems and control IIT Bombay. So we have in our background, our usual motivating image, which is essentially a rover on Mars, and these sort of vehicles are driven autonomously. Because it is not usually possible to have, sort of humans controlling the equipment at all times, on these extra terrestrial locations. So, we want to be able to, of course, analyze and learn to develop algorithms that are going to drive systems such as these.

So, until last time, what we were doing was essentially discussing the notion of positive definiteness. So this is what we were doing until last time, we were talking about positive definite functions. So this was 1 of the building blocks in the Lyapunov theorems. So we talked about positive, what is the positive definite functions. And after that, we sort of wanted to look at a few easier tests for positive definiteness. So, we saw the definition, but then it is not always easy to apply the definition.

And, so we wanted to look at some additional tests. And we looked at these, easier sort of tests that can be applied. And we sort of also saw the connection between positive definite matrices, and positive definite functions through an example. And so through this example, and of course, we saw, another example for positive definite function also. So we sort of go ahead in this way of discussion.

And the next sort of function is a radially unbounded function. So let us sort of start our lecture 4.2 here, so we are starting to talk about the radially unbounded functions. So radially, unbounded functions are sort of the next level of functions. As you can see, you have scalar valued continuous function again, of course, this should be R I mean, I guess we use R here too.

Let me see. We are using R here, so I guess we can continue to use R here, although R plus would suffice in all these cases. We do not necessarily have to use R itself. But R plus is good enough. So, non negative reals for time is usually what we have, So we are of course, looking at a function of time and a space variable, which maps to real numbers. This has always been the setup for these, functions that we denote as V which are essentially going to be our Lyapunov functions subsequently.

So, what do we require? We require that the function be 0 at 0, So at 0 value of state, the function has to take the 0 value. So this was the same condition for positive definiteness also. The big difference happens here, we want the existence of a class KR function, which this function V t, x dominates. So, in the case of positive definiteness, this was a class K function. So, what is the difference between a class K and a class KR function?

We have, in fact seen several examples. So, if you have a class K functions, it is it can be something like we are allowed to have something like this, function which sort of grows grows, grows but stays bounded below this. So, it is always increasing but stays bounded below this on the other hand class K infinity function has to be something like this I mean, which keeps drawing and goes to infinity. As t goes to infinity.

So, here of course, here, here you have the x and you have sort of phi x here and since, of course, these functions are take as arguments non negative real so, obviously, you use a norm this is what we have been doing. So, in just to compare again with the definition of positive definiteness because the same sort of expression phi norm of x the only difference being that here it was a class K function and now, in the case of radial unboundedness we require V to dominate a class KR function. So, this naturally sort of implies that this domination has to be for all time.

However, notice that just like before we do not require if you think about it, we do not require the V to be a continuously increasing function like a Class KR or function is of course, always increasing function always strictly increasing function, but all we want for V is to dominate this class K function. And notice that again unlike the definition for positive definiteness here, this lower bounding has to happen for all x in Rn.

In case of positive definiteness it was sufficient if it happened for x in some ball. Therefore, we had this nice picture where beyond this ball the V was allowed to drop below the class K function. So, essentially the idea is that it can give us local stability properties So, the idea of positive definiteness being defined in this manner is that it helps us conclude local stability properties.

On the other hand, radially bounded is typically used to conclude global stability properties and so, we require this function V to dominate this class KR function which is this guy in this particular case here for all time and for all values of the state.

So, this is much more stricter condition as you can imagine, however, this still does not require that my function V be increasing or anything it can be oscillating like this, as long as it is bounded above just a second, as long as it is bounded above your class KR function, you are fine. So, this is sort of an example this picture sort of gives you an example. The only thing is there is no crossing happening anymore, it always has to lie above this class KR curve.

And in such cases, V is said to be radially unbounded and just as we already stated before, radial unboundedness is related to notions of global stability, just like positive definiteness is connected to local stability, radial unboundedness is connected to global stability.

What would be examples, very simple examples, I will say I have x in R2 and denote it as x1 comma x2. So, V x1, x2 equals to x1 squared plus x2 squared by 2 is in fact radially unbounded function. Why is it radially unbounded? Because if you look at this function itself, this in itself is a class KR function this in itself is a class KR function. Why? Because it is 0 at 0. If you plugged if you plug 0 value of the state that is x1 and x2 are both 0, which is what it means for x to be 0, I get a 0 here and this is strictly increasing and going to infinity as any x goes to infinity. So in any direction, if you go to infinity, this is going to go to infinity.

So therefore, this is itself a class KR function. Therefore, V is in fact that radial unbounded function also every class KR function session will also be radially unbounded function. Of course, I have I can have slightly more involved examples like I can take something like V with t also x1 x2 as t by 2 x1 square plus x2 square and here of course, t is greater than equal to 0.

So, here of course, this is greater than or equal to of half x1 square plus let me be careful here, let me say this is not t but this is t plus 1. So, this is greater than equal to half x1 square plus x2 square because t plus 1 is greater than equal to 1. So, therefore, this is greater than equal to half x1 square plus x2 square, so implies V is again radially unbounded.

So, we can also have, let us see, let me try to think we can also have some kind of oscillating examples like the one that we showed in the picture. And that can be constructed say by again introducing a time variable as 1 plus sine square t divided by 2 x1 square plus x2 square. So, this is again greater than equal to half x1 squared plus x2 squared because sine squared t this is of course, greater than equal to 0. So, this is this is fine, and this is radially unbounded.

But, what this term is going to do? This, what this term is expected to do is to sort of make things oscillate around a bit. Make things oscillate around a bit. So, this this term is going to contribute to some oscillation, but in time, here, if you notice the oscillation was with the state itself that can also be constructed. I am not doing it here but this can that can also be constructed. So, I would really urge you to give it a thought as to try to try to construct such a Vx.

Trying to construct such a Vx where it is, x and it is oscillating with respect to x itself, but it is still dominating a class K function, still dominating a class K function. It is not too difficult, honestly, speaking, it is not too difficult. I mean, you can think of, I mean, the simple ideas are like, and I can add an oscillatory signal to x square. Yeah, something simple like that. As long as I do that, I am also still okay.

So, these are all examples of radial unbounded functions. Again, relatively simple examples. But, again, when things become more complicated, we do require easier conditions just like before, just like the case of positive definiteness we do require easier conditions to verify radial unboundedness also. So, what is it? So, again, if I if I think of the case when V is not a function of state, and remember that when V is not an explicit function of time, and remember, whenever V is only a function of state, we have been using Wx as the notation do not get confused at all. And using V and W almost interchangeably, just the purpose of the function is what is important than the notation of the function itself.

So, whenever V does not depend on time explicitly, I am denoting it with Wx. So, whenever it is just a function of the state, then I have three requirements. First is that W 0 has to be 0. Wx has to be strictly positive for all non-zero states. Again, I will come to the difference between the positive definiteness case soon.

And finally Wx has to go to infinity as the state goes to infinity. So, this is important, it has to go to infinity in any possible direction has to go to infinity in any possible direction. So what is the difference? First of all, let us try to see what is the difference from the easier condition for positive definite.

So, let us keep this in mind. And, remember that this third condition did not even exist for positive definiteness, there was no requirement for the function to go to infinity at all. So obviously, this third condition is completely distinct. So, we are not even going to discuss it, let us look at the first two conditions only.

If you look at the first two conditions, you require the same first condition, the function itself be 0 when the state is 0, there is no difference that this is pretty much uniform along all definitions concerning Lyapunov functions, and that the function needs to be 0 at 0. Why? Because 0 is where we are interested to go to. So, think of the Lyapunov functions or these V functions as some kind of energy functions. And we want the system to settle at the origin that is x equal to 0, therefore, we want the energy so called energy or this notional energy function to also be 0, called minimum value at 0 at the origin. Does not make sense otherwise. If it is not at the minimum value, when it is at the origin, then it could possibly go lower. And therefore, we will move from the origin does not make sense, since we want to stay at the origin. So all these functions invariably require that they be 0 when the state is 0, so this condition is identical, no difference.

The second condition is where there is a small change, it is positive for all x, but in a ball removing the origin it only has to hold true in a ball.

Of course, removing the origin, but here we need it to happen for all Rn. This is the also because how radially unboundedness is defined, that this domination has to happen for all Rn. Everything has to happen for all Rn, because we are talking global notions here. So this is the difference.

So the first two conditions, If I may I would like to call it some kind of global, global positive definiteness. It is still positive definiteness. And the only thing is we have, we are requiring something more stringent that this positiveness hold for every value of state other than the origin and not just in a ball around the origin. So this is a little bit more stringent.

And beyond that, we require that the function goes to infinity as the state goes to infinity. So what we want to do to look at this in a sort of easier condition thing, is to look at a lot of counter examples. So and that really helps us to understand what is not radially unbounded function. So I am going to label this counter examples. And 1 V apologize V x1 x2 is equal to half x1 plus x2 square.

So the question is, is this positive definite? Is this radially unbounded? So, you have to answer all these questions. So the first thing that you need to notice is that along x2 equals minus x1. What is this? If I look at the states space. I am going to draw the state space. This is x1, this is x2, why that is happening, this is x1 this is x2 and what is x2 is minus x1. I believe it is something like a line which looks like this. This is, this x2 is minus x1 is this line.

So, this is x2 equals to minus x1. So, what happens? Along this line along this line, so, I cannot of course draw it, because it is, I like to do it in three dimensions, V takes 0 value so, V equal to 0 all along this line. And that is a problem, that is a problem. Why? Because since V not greater than 0 for all x1, x2 not equal to 0.

In fact, this is a line, which can go all the way from 0 to infinity and minus infinity, where V is in fact, not positive at all. It is in fact exactly 0. Therefore, V is not even satisfying the first two conditions. So, this is, you can you can really safely say that not positive, definite even. So, forget radial unboundedness this function is not even positive yet. So, definitely does not work.

Next counter example. So, I am, I should have use W but that is fine. I used to V, but again, let us remember we are using V and W interchangeably, and it is the purpose that it serves is what is valuable to us and not the notation. So, let us not get too confused or too hung up with the notation all.

So let us look at V, W, now, I will use W, x1 x2 is equal to half x1 square plus 1 fourth, x1 4. Is this positive definite? Does it satisfy? So of course, W0 is 0. But, what happens let us W of all 0 comma alpha is also 0. Because notice that is no x2 here, only 1 state appears. So whenever your function which has only one state, what happens I can just say W0 alpha is equal to 0, which implies W not positive definite. And if it is not positive definite implies W not radially unbounded. Because I, I need minimum positive definiteness to even go to radial unboundedness. So no, answer is no again.

So, if you have any function, which has only some of the states appearing and not all of the states, it is immediately not a positive definite function and therefore, not radially unbounded. So, you do not even have to do any calculation. Because you can always make an argument like I made here.

Let us look at a slightly better example. W x1 x2, well I, let me something better. We have already seen this example. Wx. where x can be Rn. So, what about this? Does this satisfy the first two conditions? One is, is 0 and 0? Yes. W0 is in fact 0. What else? Wx is in fact positive for all x not equal to 0. It is obvious because if x is not 0, norm x is positive. We already made this argument. So, so this immediately implies norm x is positive. And I am done.

If norm x is positive, then this is positive. This is again, some positive denominators, so it does not matter. So Wx is positive. So of course, it satisfies the first two conditions. We are happy at least we have something that is positive definite, which we already proved had did prove in the previous lecture. So there is not a different example. What about radial unbounded that is what about the third condition? Does it go to infinity as x goes to infinity?

As you can see, the answer is no. Why? This is where I had written a sort of expression for W last time, this is where it will come to be of use to us. So, Wx can be written as 1 minus 1 over 1 plus norm x square. So, as x goes to infinity, Wx goes to 1. And because as x goes to infinity, this goes to infinity. So this whole thing goes to 0. And so I am left with this one. So, this is one of those functions, which is a nice positive definite function. So, so from here, I of course, had this is positive definite, but because of this implies W, not radially unbounded RUB is the notation for radial unboundedness.

So it is a positive definite function, but it is not radially unbounded because it does not go to infinity as the state goes to infinity. So, this is rather critical, rather critical distinction. So, function like this cannot be used to prove global stability. It will not help us to do global stability properties that we will see later on of course. So, this is not radially unbounded function. So, these are sort of the easier condition corresponding to radial unboundedness.

The second condition is when we have a function of both state and time, both time and appear. Exactly, the exact parallel to what we did for positive definiteness we had one condition for when we had a function of the state only and another condition when we had a function of both the state and the time explicitly.

In such cases, we just use the previous easier condition. We just again just like, we just use the previous condition what do we say. It still has to be 0 for 0 state values and all time. And but we said R plus here. Of course, fine this non negative time anyway to be honest, just to prevent ambiguity, which better to say it is R plus. For all definitions of V time in R plus is quit okay.

Because we never deal with non negative with negative time. So, we have V t 0 equal to 0 is obviously our standard requirement. But, the next requirement is simply using our previous positive radial unboundedness definition is that this V t, x has to dominate a radially unbounded function of the state only V t, x has to dominate Wx which is radial, which is radially unbounded function of the state of where Wx is radially unbounded.

So, Wx is a radially unbounded function of the state only. In this case of course, constructing examples are really easy. I can simply take a V, V t, x as again something like t plus 1 by 2 and norm x square and we know that this is greater than equal to half norm x. When, which is of course radially unbounded. So this implies that this is radially unbounded, it is not very difficult to construct such radial unbounded examples of both, state and time.

So, what did we look at today, we essentially did the exact same thing that we did for positive definiteness on the case of radial unboundedness which are functions which addition in addition to positive definiteness globally also go to infinity as the state goes to infinity. We looked at some easier conditions to verify this radial unboundedness also. So, you can of course, we will continue with the next set of functions in the upcoming lectures. So, that is it. Thank you for joining.