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Nonlinear Adaptive Control
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Week 1
Lecture 2
Preliminaries- Part 1

Welcome to another session of the NPTEL on Nonlinear and adaptive control, I am Srikant Sukumar from Systems And Control IIT Bombay, you folks can again see this very nice representative background image that has been ubiquitous in our course, this is sort of amalgamation of sensors, robotics, actuation and algorithms, the sort of algorithms, which we seek to design that will drive autonomous rovers, such as these on Mars and Moon, and so on and so forth.

So, without further ado, we go into our lectures. So, until last time, we had sort of looked at an introduction of adaptive control, we sort of saw what these building blocks mean, there is a state space model block, there is a controller block, but in addition, there is also an adaptive control block.

So, the purpose of this adaptive control block was to estimate these unknown parameters in the system. So, now as we move forward, one of the things that we sort of need to remember is that the purpose of or a lot of what we do in adaptive control and nonlinear control in general is this asymptotic analysis.

So, that is we sort of try to see what happens, as time goes to infinity to different signals and functions, and so on and so forth. So, this is one of the key aspects of what we are going to do, as we move further along in this course. Now, there are a few pitfalls in this sort of journey of ours. Now, one of the, first few pitfalls is regarding, what happens to the convergence of functions, and correspondingly what happens to the convergence of its derivatives.

So, this is what we want to address first today in this lecture. So, suppose I consider a function and let us see, if I consider a function and I have this knowledge that as t goes to infinity, the function converges to a constant. So, suppose I know that the function converges to a constant as t goes to infinity, this is what we are talking about, then this does not automatically imply that the derivative of the function goes to 0. Intuitively, we sort of assume that, this should be true that, if a function is converging to a constant, then the derivative goes to 0, if a function goes to 0, the derivative goes to 0, all of this somehow seems to make sense in our minds. But this is not true.

This is one of the first pitfalls. So, we quote a counter example. Of course, there are many, many counter examples possible, we quote one of them. And what is it? Suppose I take the function, which is sine t squared divided by t. And suppose I take the limit of this function as t goes to infinity. Let us look carefully at what happens, the numerator keeps oscillating between minus one and one, it does not matter what happens to time and how much time is lapsed?

If I plug in something like an infinity or a very, very large number here, it does not matter, sine of t square is still going to oscillate between minus 1 and 1. So, this is not really reaching a limit, however, the denominator is going to go to infinity. As t becomes really, really large, the denominator is going to become very, very large and therefore, it is obvious that the denominator is becoming very large when the numerator is anything between minus 1 to 1, so it does not matter what happens to the numerator, as I take a limit as t goes to infinity, we get 0.

So, we get, we get that our function converges to a constant which is 0. So, this is what, it is one of the things that we need to remember. Now, let us look at the derivative of this function. So, what is the derivative? It is very standard we just computed using the chain rule and the product rule. So, the first piece here is the derivative corresponding to the 1 over t term here.

And the second term here is the derivative corresponding to the numerator it is very easy to compute. I really encourage you to do this computation by yourself not to believe me. Then what happens? Now, we look at what happens to each of the term if I take a limit on both sides.

What happens? Now, if I look at the first term, again, it starts to look very much like this term there is a sine t squared minus sine t squared, which is again, oscillating. So, something between plus minus 1, but the denominator is again, going to go to infinity, in fact, faster than the 1 by t.

Which means that this term is definitely going to go to 0. So, this term is definitely going to go to 0. No problem, what happens to the second term, though, what we note that the second term now does not contain anything in the denominator, it is only a trigonometric function that is a cosine t squared in the numerator, which is not so great anymore.

Because now I know that the cosine t square is just going to keep oscillating and not reach a limit at all. And this is something which should bother us to be honest. Because honestly, speaking, this function is not that is f dot of t, in fact, will not have any limit as t goes to infinity.

So, the derivative of the function has no limit as t goes to infinity. So, this, whatever seemingly counterintuitive thing has come true. That though the function itself converges to a constant 0, in this case, the derivative does not really converge to anything at all. And this is, should be baffling for us.

The important thing to notice that we did not really take any very badly behaved function, it is not a non smooth function, it is not a discontinuous function or anything like that, except at t equal to 0. The function is, infact, C infinity, what is C infinity, C infinity means infinitely times continuously differentiable.

So, this is infinitely times continuously differentiable, except at t equal to 0. For those who have never seen this notation, C infinity, I strongly encourage you to look up this notation. Make sure you familiarize yourself with it, because this notation is definitely going to show up regularly in our course.

So, the important thing to note is we did not choose any poorly behaved function to sort of prove our, thesis that the function converging to a constant does not mean the derivative converges to a 0. We picked a relatively nice function. So, the derivative in fact, continues to oscillate between minus 2 and 2.

So, that is what we say here. So, the derivative does not go to 0, but it oscillates between minus 2 and 2. So, the converse also does not hold true, interestingly. So, what is the converse? The converse is that if the derivative goes to 0, can we say that the function itself is going to go to a constant?

So, this is again another important question. What is the question? If the derivative goes to 0? Is a function itself going to go to a constant? We again present our hypothesis and say that no, this is not true. And again, we present some counter example. Please ignore this. I have deliberately removed, cancel this because this is erroneous, this is wrong, but look at this example here. If I take a function f of t, which is log of t, logarithm of t, and the derivative is 1 over t, we all know what is the derivative of the logarithmic function simply 1 over t.

And we know very well that if I take the limit as t goes to infinity for f dot of t, then it is 0. Because the t goes to the denominator goes to infinity and therefore the limit is 0. On the other hand, if I take the limit of f of t, bad things happen, because t goes to infinity and logarithm, of course, goes to infinity, maybe slower than t but still, it does go to infinity.

So, therefore, limit of f of t is in fact infinity. That's bad, in fact, it is not a bounded function at all. However, the derivative is actually going to go to 0. Again, if you note, this function is nicely behaved everywhere except at the origin. Except at the origin, this function is nicely behaved. So, remember that we are only sort of interested in the behavior, at least as far as asymptotic analysis is concerned at infinity.

And that is at very very large time. So, you are not really interested in small time behavior. So, therefore, the function not being well defined at the origin and so on should not matter so much I can always tweak the function so that it is well behaved at the origin. And everything goes through how would I do that? Very simple, I will say, make this sort instead log t, I will make log t plus 1.

And now, the functions derivative is 1 over t plus 1 and I still have the exact same result, that is one over t plus 1 goes to 0, as t goes to infinity, log t plus 1 blows up as t goes to infinity. In fact, this is now very well behaved very well behaved at the origin. And since the time is always considered to be non-negative, that it starts at 0. So, I have no problem, this is a very valid nice function in the domain of my interest.

So, I have a very nice function f of t, nicely behaved at the origin and everywhere else beyond t equal to 0. And still, it does not satisfy this sort of intuitive idea, may be that if the derivative converges to 0, and the function itself converges to a constant. So, I really want all of you to drive this notion out of your minds that if you have a derivative converging to 0, then the function goes to a constant and vice versa, that if the function is converging to a constant, then the derivative goes to 0.

Where this lacunae happens is because we are not looking at the function itself. We are not really looking at the function itself, we are looking at the limit of a function. We are not saying that the function is a constant, it is obvious that if the function is a constant, the derivative is 0 and vice versa.

That is obvious, but we are not saying it, we are saying that if the function converges to a constant, so the limit in these, in all our discussion, this limit is what somehow messes up things for you. So, this is very critical, please keep this in mind. So, I really strongly, strongly encourage all of you to come up with more counter examples. I know you can, so give it a shot. Just take hints from what I have done, the more you sort of come up with counter examples, the more you will convince yourself and also actually develop the habit of coming up with counter examples.

In applied mathematics and mathematics, coming up with counter examples is one of the most challenging things, most mathematicians, a lot of mathematics actually write papers out of coming up with counter examples of some other results that somebody seems to have proven, just to claim that, that result is probably not correct. So, you can actually make a business out of coming up with counter examples. So, it is an interesting exercise, just I really would recommend that all of you tried to come up with these counter examples.

Once we have sort of understood and not understood, but we have sort of gotten rid of some myths, some temptations in asymptotic analysis, we come to the next important set of notions that are really really critical to what we want to study. And what is this, these are the notions of norms, there are 2 things we are really interested in one is that the fact that we do asymptotic analysis means that limits and convergence and these notions become really important for us.

And second is because we are always dealing with vectors, signals, vectors and signals, because all the states all the control outputs, everything is a vector. So, we are always working in some kind of a vector space, at least for the purposes of this course, these are vector spaces. There may be other more complicated structures in other courses, but right in this course, we are dealing with vector spaces and hence all the objects that we are dealing with are also vectors.

Since we have vectors, and then we have matrices which are operating on these vectors. It is important for us to sort of have a measure of the size of these vectors, because we eventually want to a lot of these vectors to go to 0. Or we want one vector to dominate another vector. So, to actually have a sense of how these notions can be created, like in real numbers, it is very easy to say that 3 is greater than 2.

But if I gave you a vector, for example, if I just said, I have a vector 3, 1, and minus 1, 5, can you tell me which one is greater than which one is less than the other? There is no obvious way. There is no obvious way of doing these things. So, Norm actually gives you one such obvious way because it reduces any vector to a scalar.

I am not saying that this is the way, but this is in particular way and this, in fact, gives us a lot of lot of ability to do a lot of mathematics with vector spaces. So, like I said, well, I did not say it yet. But the norm is essentially a measure of length. In a vector space, a norm is a measure of length in a vector space.

So, keep this in mind. But of course, we have a more formal definition of what a norm is. So, Norm is a function, in our case, from Rn to R. So, we are keeping it a little bit simple. So, we are only working with real vector space. So, Norm is a function from Rn to R. And it is a valid norm, if it is non-negative. So, this is the notation the 2 vertical brackets, this is the notation get used to this.

If you have not seen this before, I would expect most of you have seen some version of this. But, so, the first property that is that it is non-negative. The second is that it is 0 only if the vector itself is the 0 vector. The third is the scalar multiplication property that is the norm of Alpha x is actually equal to mod of Alpha times norm of x for any scalar Alpha and the final and the probably the most critical properties the triangle inequality, which is that norm of x plus y is less than equal to norm x plus norm y.

So, these are the standard properties for norms, any function that you can come up with that map's from Rn that is your vector space to real numbers to scalars with these properties will be a norm a valid norm, will be a valid norm. Now, what are examples of a valid norm, one is the commonly used norms are the infinity norm, which is just the maximum of all the elements.

So, if you just take the max of all the elements of a vector, then you have the infinity norm for the vector, and then you have the p-norm, what is the p-norm? The p-norm is simply taking the absolute value to the power p of each component, summing them up and then taking the pth root, so, that is what is the p-norm. Now, I am going to go forward and then back again.

So, these are rather important notions and these are other important norms. So, if for example, if I take a vector, say something like this. What is that? This is essentially a vector x in R 4, which is 3, 2, 7, 5 so it is 4 elements R 4. Now, suppose I want to compute these norms. So, what will be the infinity norm? Infinity norm is just the largest component. So, it should be obvious to you that this is the largest component, therefore, the infinity norm is 7.

What about some p-norms, there are some useful p-norms, the 2 norm is actually the Euclidean distance, like I said, what you all of you are used to in measuring distance is the Euclidean distance is the actually the 2 norm. How do you compute it? You take the absolute value of every element? Well, the example I have chosen, everything is positive.

So, absolute value does not change anything. I take the pth power, that is the squared in this case, and then I take a sum, and then I take a square root, so I am not actually showing what the answer is, but it should be pretty straight forward for you to compute. Finally, what is the 1-norm, the 1-norm is again, very similar to compute, you take the absolute value, add them up, take the first root, which is basically do nothing and then you get the answer 3 plus 2 plus 7 plus 5, which is just 17.

So, this is some key vector norms. And these are sort of some of the ideas and these are some of the examples of norms that you have. That will be very useful as we go forward. We also have the notion of a matrix induced norm. So, like I said, we have vectors and then we have matrices. So, we since, we have both vectors and then matrices which operate on these vectors.

Now, we will see how do not worry about it now. But the point is we also have to define a norm. So, the way we do this is that we use the vector norm itself to somehow generate a matrix. So, this is again something very normal in mathematics is that, whenever you want to solve a new problem, you try to get motivated by a previous problem that has already been solved or a simple problem that is already been solved.

So, here we already have the notion of a vector norm. So, a typical mathematician would think, why not use the vector norm itself to generate a matrix now? Smart. So, what is it? So, how do I do that, so, the matrix induced norm, that is why it is called the induced norm, because it is induced by a vector norm is simply measuring a maximum magnification of a vector of any vector by a matrix.

So, that is why you see this definition, it says its supremum, which is a generalization of the maximum? So, the supremum is a generalization of the maximum. And so, it is taking the supremum over all x in Rn, So, it is the supremum of all possible vectors. And the numerator is the norm of Ax and the denominator is norm of x.

So, it is somehow measuring the magnification due to the matrix A of all possible vector, vectors in the vector space. So, so, it should it is also important to notice that this matrix A is not a square matrix not necessarily a square matrix you can define a norm for any matrix it does not have to be a square matrix.

Now, the interesting thing to note is that the numerator is therefore a different sized vector and the denominator is a different sized vector, but only because of the norm again, you should sort of become conversant with the power of the norm only because of the norm, you are able to sort of compare vectors in 2 different vector spaces because R 2 and R 4 for example, are 2 different vector spaces, but just because you have this norm operation, you are able to compare the 2.

So, if you know if you have some idea of Eigen values and Eigen vectors, you can make some guesses as to what will be the maximum magnification and things like that. So, I will leave you to say to try to figure out what would be this maximum magnification of a matrix, so, the important thing to notice that, this, depending on what norm vector norm I choose here, so, if I put a p here, the p here, I get a corresponding p matrix norm, so, depending on what vector norm I use, of course, I have to use the same vector norm in the numerator and denominator, then I get a corresponding p induced norm. It is as simple as that.

So, now, before sort of discussing what is the supremum, and so on and so forth? We sort of want to understand what is the structure that is given to a vector space by a norm. So, this is something that is critical, there is something that is very critical.

So, I am of course, assuming that all of you have seen vector spaces are some variant of it. If you have not again, this is something that you really should see in linear systems course, with the state space linear systems course. Because whenever we talk linear systems, we are saying that the system itself is evolving on a vector space. The idea the notion is the idea of spaces subspaces planes and so on planes and sub planes hyperplanes so, that is vector spaces essentially generalization of these hyperplanes .

So, these are spaces where superposition principle is satisfied. So, again, I am being very vague about it, but I do would expect that all of you do know what is a vector space because otherwise you do not follow what is the Normed linear space. Whenever I say linear space, linear space and vector space are identical linear space linear vector space vector space these are used almost analogously.

So, what is the Normed Linear space? So, Normed linear space is essentially a linear vector space with an associated norm, that is it, if you have a vector space or a linear space or a linear vector space, like I said, and you have a norm on it, then the 2 together denoted as x comma this norm, form a Normed linear space.

Now, the good thing is most of the spaces we are working with, which is R k's , R p's and so on and so forth, they are all Normed linear spaces. So, basically any Rn with any of the aforementioned vector norms, that is the x infinity x1, x2 that is Rn within infinity norm Rn with one norm Rn with 2 norm, these are all Normed linear spaces, so, just the fact that you are able to define a norm on this space, makes it a normed linear space.

So, remember this, so, this is a rather nice set of notions, the notion of a Normed linear space, we will continue to use this on a regular basis. So, again, so what is it that we are looking at is some notions on sort of pitfalls in asymptotic analysis that we want to avoid? That is the first set of things that we did look at today. The next set of ideas is on the fact that we want to deal with vectors, because states controls, outputs, everything are vectors. And we want to look at sizes of these, we want to look at what happens to the size of these vectors as time evolves.

So, therefore, in order to in order to sort of make sense of all these notions and compare 2 vectors, if you want to right, we define vector norms, matrix norms also are something which are very critical, because these matrices will eventually operate on these vectors. And we want to sort of assess how this operation affects the vector.

So, therefore, we have taken the liberty of defining a matrix norm, I am using a vector norm, and this is called the induced norm. And this matrix induced norm also has some nice properties, which we will look at next time. And then we also saw that these norms on a vector space, in fact, give us a rather nice structure as of now, we are simply making definitions out of it.

But the fact that you have a Normed linear space is a rather serious object that you have all, it is a serious amount of structure that you have in this space. first of all, you already have a vector space, which is linear in the sense that you are in hyperplanes and there is a superposition principle that is followed on top of it, you have a notion of a distance on this vector space, which becomes really, really useful in all sorts of analysis, all sorts of mathematical analysis that we are very, very keen on exploring.

Even the matrix induced norm is the norm that we have developed for the matrix also make the space of matrices a Normed linear space. We do not discuss it at this stage. But this is an additional titbit for you, if you may, that these norms that we have defined for the matrix, that is the matrix induced norm also makes the space of matrices. For example, R n by n matrix along with the induced norm also gives you a Normed linear space. With that, we will stop.