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Nonlinear Adaptive Control
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Week-4
Lecture 19
Positive Definite Functions

Hello everyone, welcome to yet another session of our NPTEL on nonlinear and adaptive control, I am Srikant Sukumar from systems and control IIT Bombay. So, we are again in front of our very nice background image of this rover on Mars, which essentially is autonomously operating device and we hope that we will soon be able to analyse and design algorithms that drive systems such as these rovers.

So, what we were looking at last time was the notion of function classes, we had sort of began our discussion on the Lyapunov's direct method and leading up to it we required first the notion of function classes and then we define three classes of functions right and beyond that we started speaking about notions of definiteness, we started speaking about notions of definiteness. We had defined the first notion of positive definiteness of a function.

So, let us look at where we go from there today. So, this is lecture 4.1 now, so we are now into our fourth week lectures. So, if you look at how we define this, we define positive definiteness as requiring a couple of things first is that you have a scalar valued continuous function of course, of this form takes two arguments which is a time and the state and then we need it to be zero value for all zero states right and further we require it to dominate a class K function in some local region that is in some ball around the origin and for all the time.

So, we have these requirements, we made a very-very nice illustrative image to indicate that though it dominates a class K function it itself need not be an strictly increasing function and in fact, it can just cross over this class K function beyond a certain radius or beyond a certain bound on the states are.

So, the first thing we want to do is sort of try to connect what we learned about definiteness of matrices to definiteness of functions. So, because we are using the same terminology, one wonders whether these two are indeed connected or not. So, the first I mean, we saw that definiteness of matrices of symmetric positive definite symmetric matrices has three equivalent conditions, the first being this kind of a quadratic form, which is required to be positive for all nonzero states and then you have something like the eigenvalue conditions that is all Eigen values of A have to be strictly positive, notice that because it is a symmetric matrix all the eigenvalues are in fact real, all the Eigen values are in fact real, in fact, this is what is called a diagonalizable matrix, all symmetric matrices are in fact, diagonalizable. And further we have sorry, for the we have that this all principle minors have to have a positive determinant.

So, let us look at some sort of a function construction based on this. So, suppose, using this guy, I define a function V. So, from here, I define a function V t, x. Well, I define two functions I mean, I define it does not matter. Well, I mean, why do not I do this? I define V t, x as say t times x transpose Ax, I just in fact if you notice all I did was I took the same quadratic form right here, I take this. So, one thing is obvious that this is continuous, it is taking both state and time and mapping it to a real number because this is a real number and A is an n by n matrix.

So, the question that we want to ask ourselves is that is A positive definite matrix, if A positive definite matrix, the question we want to ask is, is the function B also positive definite? The quick answer is yes. So, I do not want to keep any suspense anyway it should be obvious to a lot of you and this is in fact the case. So, the first thing I can see is that this is greater than equal to, let me be careful here, because this t, in fact, I made this example here two.

Actually, in this example too, I think I said t but I think it is better if I take e to the power t, because if I take t is not necessarily greater than equal to this class K function because at t equals 0 there is a problem, t equal to 0, there is a problem, or else of course, I can in this case I can do something simpler, I will define this as t plus 1, this is greater than equal to x transpose Ax for all t greater than equal to 0. So, I have this to be true. So, I have already have like a time independent function on the right hand side, the only question I want to ask is whether this is a class K function or not.

Now, one thing I know for sure is that for a symmetric matrix A, I can write this as some y transpose lambda y, where I apologise where you have this lambda is the diagonal matrix of eigenvalues lambda is the diagonal matrix of eigenvalues. So, this should be again this should be something that is evident to us that lambda is the this is basically the standard Jordan decomposition for a symmetric matrix. So, lambda here is a diagonal matrix of eigenvalues of the matrix A.

So, what do I know about x transpose Ax right. So, this guy x transpose Ax is simply x transpose y transpose lambda y x. So, this is, so, this is basically yx transpose, sorry let me be careful here this is not y, I apologize, this is not some vector y this is not some vector y this is actually equal to some matrix, some M is Eigen vectors of A eigenvector matrix it is not this is not quite. Let me redo this.

So, this is x transpose M transpose lambda M x and that is basically Mx transpose lambda Mx. So, M is of course inverse is equal to M transpose in fact, M inverse this M transpose for the symmetric matrix case. So, the inverse actually exists and it is in fact equal to the transpose. So, you have Mx transpose lambda Mx. So, you know that what do I know? So, I know that this is basically some kind of a function. So, let me call y equal to Mx then this is equal to y transpose lambda y, which is simply summation over i lambda i Yi squared, where lambda i's where these small lambda i's are eigen values of A.

Now, because A is positive definite, I know that lambda i is a positive. So, this is basically what can I say? I can say that this is essentially a just a quadratic summation of quadratics just like what you have in a quadratic form, this is what it means to be a quadratic form. So, basically this is something like a lambda i Yi squared. So, though not in the original states, but in some transform states, notice this transformation is very nice, because M is an invertible matrix, so very nice transformation.

And through this transformation, I get this summation lambda i Yi squared. And I claim that this is in fact a class K function in the original variables also. Because, again, I mean, it is not, I am not going to do the rest of the math here, because I have an easier test for doing positive definiteness, which I will come to soon. But my claim is that this is x transpose Ax is in fact a class K function. So, this is my claim belongs to class K.

Now, but I can already see that it is a sum of quadratics and because it is a sum of quadratics, it is also going to be a sum of quadratics in x is equivalent. And, therefore, this is and you know that the sum of quadratics in x is a class K function, basically, if I give you a function of the form x1 square plus x2 square, we know it is a class K. Great.

So, this is not complete, I know that you are not yet convinced that this is a positive definite function, but you see that, if I am given a positive definite matrix, I have positive Eigen values and I get a quadratic in a sort of a modified state, which is y equal to Mx. So, this much let us remember this much, this claim, maybe we come to later, let us not worry about that. So, the point is, there is a clear connection, we will establish that connection sooner than you think.

So, that is where we come to the easier conditions. So, one of the issues with this condition that we have, that is V t, x to dominate a class K function. One of the issues with this is that it is very difficult to verify, because you have to actually find the class K function which V dominates. Now, the other issues if you are unable to find such a class K function, that is not enough to claim that V is not positive definite. This is because you may have missed being able to find it, it could also be our own incompetence that we could not find a class K.

So, just because we could not find a class K function does not automatically imply that V is not positive definite, all we can say is that if we do find a class K function, V positive definite, but if we do not find a class K function, we cannot for sure say that it is not a positive definite function. So, we want easier conditions where we can say for certainty.

So, we have two different conditions. The first one is when the V, which we are now denoting W depends only on this state, if the function depends only on this state and not explicitly on time. So, therefore, we have used a different sort of symbol here W, but it does not matter call it V, call it W, call it Z, your call. So, this function W now depends only on the state, therefore, its argument is just Br, and it maps to real numbers with x being mapped to W x.

And it has if it satisfies two conditions, the first is that it is 0 at 0, which is the same as this guy, this is easy to verify. So, we are not really modifying this condition. Again, no time appears here, because of course, there is no time argument. So, we want the W 0 to be exactly 0. And next, we want W x to be strictly positive for all nonzero values of the states, you want W to be strictly positive for all nonzero values of the states. So, then the function is said to be positive definite.

So, let me go to the second definition, which before I go back to our matrix example again. Now, if we do have a function, which depends on both state and time, this may be unavoidable for certain dynamical systems, especially dynamical system, where the vector field also explicitly depends on time, as we have been considering in our stability definitions, it might become impossible to avoid having a tiny argument explicitly in the Lyapunov candidate construction, so this V function construction.

So, in those cases, what do we require? As usual, the first condition remains because this is easy to verify. And the second condition requires that this V t, x dominate a positive definite W. Simple, we will use the sort of use this previous guy, right here, because V is a function of the state also of the time of time also explicitly and we have no direct test for dealing with that, what do we do we say that V t, x just has to dominate a positive definite function, with just a positive definite point, we are not saying a class K function remember, we are just saying positive definite function and this is much easier, because I can verify positive definiteness with this easy tests.

And once I have that easy test satisfied, and V dominate this positive definite function, then V t, x is also said to be positive definite. And this domination of course, has to happen for all t in R, and for all x which is in in Br but 0, so this is what we need. So, these are the two conditions, of course, the notation is that V is greater than 0 is positive, but if we want to talk about negative definiteness, we just say that minus V needs to be positive definite. And in negative definite cases, we use this notation, V is less than 0. So, whenever I use a function, and say it is less than 0 greater than 0, I mean, it is positive or negative definite.

Now, let us look at our matrix example. Again. So, we want to sort of complete, I apologise discussing our matrix example. So, what was the function we took V of x as, we took v of t, x as t plus 1, x transpose. Sorry, I really apologise for them, what did I take it as? I think I took it as x transpose Ax. And the first thing I know I can do is that this is greater than or equal to W x, which is exactly equal to x transpose Ax. So, I think what I will do is I will not claim, now earlier, I said that I claim positive class K, so I will not do that anymore, because we do not really need that and I am not going to be able to directly prove it anyway. So, I am not going to claim that, what I am going to instead claim that this is in fact, positive definite.

So, let us look at we have already seen W x is x transpose Ax is actually equal to summation over i lambda i Yi squared, where Y is basically equals to Mx and A is written as M transpose lambda M. So, this is the Eigen value decomposition and this is just the Eigen value transformation, Eigen vector transformation. And of course, we know that M is invertible, we know that M is invertible and it is exactly equal to M transpose. So, M is invertible and it is exactly equal to M transpose.

So, what do I have? I have basically this sort of expression here. So, let me sort of right, make a nice big box around it always end up doing this, let us see. So, this is what we have now. Now, what do I need to show for W x to be positive definite? I need to show that W 0 is 0, that is obvious. And because if I plug x equal to 0 here, it is definitely 0 no problem. The second point is the more difficult one. I want to show that W x is strictly positive for all x not equal to 0. I am not using any Br here because this is there is no Br and in fact, I will I can even say this. I am in this case I am trying to prove that.

If I remove the origin from R n, then except for the origin everywhere as W X has to be strictly positive and it is not difficult to see that it is. Because of this guy, because of this transformation, you have that x not equal to 0 is equivalent to y not equal to 0. I hope this is obvious to you, simply because M is an invertible matrix, because M is invertible, a nonzero x is equivalent a nonzero Y and vice versa. And whenever I say 0, I am very casually putting this zero but I hope all of you are very clear with this notation. Whenever I say non-zero, it means that I am talking about the zero vector.

So, when I say x is not zero, I mean not every element of x is zero, some elements are still allowed to be zero, of course. So, here we are speaking about just a zero vector that is every element is zero and a nonzero vector means that at least some elements are nonzero, at least some not every element nonzero. There is not component wise or anything. This is exactly how it is written in typical mathematics. A nonzero vector means there are some elements for sure which are nonzero. Excellent.

So, once I know this, once I understand that x not equal to 0 implies y not equal to 0, and that is these are equivalent. So, whenever x is not 0, y is not 0. So, I can simply say from here that this implies that y, so, x not equals 0, equivalent to y not equals 0, which implies summation i lambda i Y i square not equals 0, since lambda i's are strictly positive, because I assumed A is positive definite, therefore, all lambda i's are strictly positive. So, i equal do 1 to n, if you may want to make it more precise.

So, you have that every lambda is positive, therefore, if y is not a 0 vector, there is some element which is nonzero and therefore, this cannot be 0, because each of them is contributing something there is no subtractions here, so only summations. So, once I have this argument clear, that if x is not 0, then this quadratic form that is this quadratic is nonzero, which implies that, in fact, not just zero this is in fact, I should make it more clear that this is in fact, greater than 0. It is not just not 0 is greater than 0, simply because this quantity can never be less than 0, this quadratic form can never be less than 0, it is greater than 0 or equal to 0.

Therefore, this immediately is equivalent to this. Therefore, I immediately say that W x is strictly positive if x not equal to 0, and I am done. So, W x is a positive definite function and V t, x dominates this W x, because this is just going to be more than greater than or equal to 1 and therefore, because V t, x dominates this W x and W x has been proven to be a positive definite function. Therefore, I have a positive definite V t, x . So, what have we concluded that, so remark positive definite matrices lead to positive definite function constructions. So, just in the we probably want to see at least one more example right now.

So, let us look at another example. So, let us say V t, x. In this case, I specified as x1, x2 times x1, x2, are scalars, of course. So, if I look at this as t plus 1 times in fact, why do not I just do this, I do not need this, I just use x and this as here is a norm x square divided by 1 plus norm x square. So, this is again greater than equal to norm x square divided by 1 plus norm x square. So, this is say my W x. So, this is of course, again positive definite function, this is again a positive definite function, why? You look at W 0 is 0, if so, this is the first point.

Now, if norm x not equal to 0. So, equal into x not equal to 0, x is not equal to 0 means norm of this x was also not 0. So, then, if x is norm x is not 0, you can see that W of x is strictly positive. Very easy to conclude, in fact, I can, it is even easier if you notice that W x can be written as 1 minus 1 over 1 plus norm x square.

So, x is, in fact, I do not even need to write it in this form, even in this form it is obvious. If norm x is nonzero, then this is strictly positive, this is strictly positive, this is strictly positive, but then this is just dividing this guy. So, the numerator and denominator are both strictly positive. So, this is a fraction which is strictly positive. So, I am done. So, implies of course that positive definite or V greater than 0 in a notation, great.

So, what have we talked about today, we continued our discussion on definiteness. And we connected the notion of definiteness of matrices to that of functions. And we saw that positive definite matrices yield positive definite functions. And this is a very-very standard way of constructing Lyapunov functions, in fact, so, V is essentially, we will see our Lyapunov functions called Lyapunov functions.

So, and then we of course saw these alternate tests for positive definiteness, which are easier to actually verify, and we used this to construct some examples of positive definite functions. So, we will again continue on this line a little bit more. And, yes we will, of course, this is again leading up to the Lyapunov theorems. Excellent. So, that is all for this session. Thank you and we will meet again.