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Nonlinear Adaptive Control
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Week 3
Lecture 18
Function Classes and Definiteness

Hello, everyone, welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control. I am Srikant Sukumar from Systems and Control, IIT Bombay. So, we are almost at the end of our lectures for week 3 and we have sort of learned over the course of this week, some nice tools to analyze algorithms that will potentially drive systems such as that we see in our background.

So, what we were doing until last time was basically look at a few more examples of these particular stability properties that we had defined. So, we saw few more illustrative examples. So, I really hope that all of you got a pretty fair idea of what kind of systems are going to have, what kind of properties and what do these properties sort of mean for that particular system. So, we defined stability, attractivity, and several notions of asymptotic and exponential stability.

We also saw at the end of the last lecture, the particular notion of linear system stability. So, basically how does the stability definitions in terms of epsilon and delta's that we defined, how does it become specific, more specific for linear systems. So, we sort of had a brief overview of that, of course, we did not really prove these results as such, so, I just gave you a little bit of a sketch of what you have in linear system stability.

So, today, we jump on to a new set of lectures again, like I said, do not worry about when we say week 4 and so on, although we are not into week 4 lectures, this is more for the purpose of the homework assignments. So, we did not sort of go forward and delve into the week 4 lectures, depending on how our schedule is, because as the lectures become more involved, we will need more and more time.

So, therefore, I do not want to delay getting into week 4 lectures any further. So, as the outline states, we discuss function Classes and Lyapunov theorems, in this particular set of lectures, this was designated week 4 lectures. So, we have already start. So, now, what is the motivation? What is the motivation? So, we already saw several stability definitions, we saw a few examples where we did conclude what kind of stability properties a system might have.

However, what we found was that, in all these cases, we were required to find the solution, or in some cases, of course, we made a call on stability based on phase plane portraits, but to be honest, that cannot be considered final sort of, final technical answer on whether a system is stable or asymptotically, stable or not, because a lot of times in phase plane portrait, again, a phase plane portrait is based on a numerical method, you sort of take a software and you draw these phase plane portrait changes based on a numerical method.

So, it is very, very, very possible in numerical methods to ignore or forget to consider certain problematic initial conditions, and you might actually end up claiming something about the stability of a system which actually does not hold true. So, the biggest issues with these epsilon delta definitions of stability is the usability. It is very difficult to actually use these definitions.

So, that is the motivation for studying these Lyapunov theorems that we are going to look at now. So, Lyapunov theorems really make our life easy in terms of concluding stability of nonlinear systems and we will very quickly get to that, but before we even start on to Lyapunov stability theorems or what is also called the Lyapunov, direct method you will of course need to define a few really preliminary Class of functions.

So, these are rather powerful Class of functions and so, there are 3. So, the first is called the Class K function. So, what is a Class K function, a function again. So, these are all scalar valued functions, so, they take values in non-negative reals and map to non negative reals. So, this is required for all these functions. So, they take values in non negative reals and they map it to also negatively reals, so, a function fine of this kind is said to be of Class K, if it satisfies three properties first it is continuous, second, it is strictly increasing. And third, it is 0 at 0 to 0 at 0. So, this is what is Class K function.

So, we are of course, given examples of what is the Class K function. So, if f x equal to x, so let us be a little bit more precise. So, say examples and sort of write it again. F(x) or let us use the notation that we are in fact using here for x in non negative reals is a Class K function or if you want to make it, much more easy to understand.

So, Phi x is x squared with x and R plus is also Class K function. So, notice that I keep saying x is an R plus, because I can either way I have defined in this map has to be from non negative reals to non negative reals. The argument also has to be a non negative real number. So, this is important. The next example we consider is, I do not want to number these because I have sort of given a few different examples.

The next example that I gave, that is seated here is 1 e minus x. But again x in R plus. So, how do I know that this is sorry, this is 1 e minus x not 1 e to the 1 minus e to the power minus x. How do I know this is a Class K function? First of all, continuity is obvious. How do I know it is a Class K function, just compute phi prime of x, maybe it is simply going to be e to the power minus x which is always greater than 0 for all x less than infinity.

So, which is basically saying that x is in R plus. Non-negative reals does not include infinity. Infinity is not included in non negative reals, or in reals for that matter. So, that is how real numbers are defined on it. So, this is of course, you can see that this is a positive derivative and therefore it is obvious that it is a strictly increasing function. So, you can have many different such examples. So, this is what is a Class K function.

So, what is the Class L function? It is sort of the opposite, it is sort of the opposite. A function phi is Class L, again with the same domain and range if it is again continuous strictly decreasing and the initial condition has to be finite, some finite initial condition. So, one example again of such a function is 1 over x plus 1. So, this is a Class L function, this is continuous because again this is assuming x is in R plus. It we are already assuming that x is in R plus.

So, therefore non-negative reals therefore x can the least value x can take 0, that is the denominator is non-zero there. So, it is in fact a continuous function, it is a continuous function. So, it is continuous, it is strictly decreasing is obvious again, because as I increase x, the denominator is becoming larger and therefore the fraction is becoming smaller then the initial value is of course finite in fact the initial value and that is, if you plug x equals 0, it is just 1. So, this is the Class L function.

So, we have Class K, and then Class L, it is sort, seems like it is opposite but I would in fact, put a nice note here. If Phi is in Class K is the notation for saying that a function is in a particular Class, this does not imply, I hope you understand that minus Phi in Class L. That this does not apply this, that is how. I hope that is clear. Because why? Because just look at minus Phi becomes a map from R plus to R minus.

So, this is not even allowed. So, this is not even allowed, not allowed. The only candidates that are allowed are infact your functions which are mapping from non negative reals to non negative reals. So, negative of Phi is not a Class L function if phi is a Class K function, excellent, excellent, great. So, good, so, we know that.

Let us look at the next Class of functions. Again, another very important Class of functions. And that is the Class KR function. This is the Class KR function. So, function Phi, again, non negative reals to non negative reals, let us always remember that, so we do not make a mistake. So, this is of Class KR. If again, it is at least Class K, first of all, you want it to be Class K and further, Phi of R goes to infinity as R goes to infinity.

So, it is first of all, it is Class K, and it should go to infinity as R goes to infinity. So, just to distinguish one of the examples, we consider it as a Class K function Phi x equals 1 minus e minus x with x in non-negative reals is not Class KR. Why if you sort of try to make a picture of this function. So, if you make a plot of this what this function looks like. What happens on the x-axis I have of course x itself and on the y-axis I have Phi of x, Phi of 0 as you can see is exactly 1 as of Phi of 0 is exactly 0. And I will what I will do is I will draw the line corresponding to 1 because as x goes to infinity minus x goes to minus infinity this goes to 0.

So, Phi tends to 1 as x goes to infinity. So, the thing is the function is in fact increasing if I may I mean of course, I cannot do justice to this very much the Phi continues to increase, but it hits at most 1 never crossses 1, never crosses 1. So, this is not a Class KR function. So, this is a rather critical distinction. So, this is not a Class KR function.

On the other hand, the other examples that we considered like Phi x equals x squared, x is an R plus. So, this Phi is of course, Class KR. So, it should be obvious to you that any function which is. So, again this is a note just so note Phi in Class KR implies Phi in Class K. So, if a function is in Class KR then it belongs to Class K for sure, but it does not hold otherwise the other way around does not hold because we just saw this example which is a Class K function so, this is a Class K function, let me reiterate that.

So, Phi is in Class K and not Class KR. So, hold this way not the other way. So, Class KR function is a stronger, it is a stronger requirement. So, there are less number of functions in Class KR than Class K. So, Class KR is actually subset of the Class K functions because Class K is a requirement of being a Class KR function. So, these are of course, rather important function Classes.

But, anyway this this we already mentioned last time too, we did talk about this in the earlier lectures also we do require that our equilibrium plays at the origin if not as always we do a change of coordinates we have already spoke about this again last time and the time before so, whenever we have an equilibrium which is non 0 we just do a simple change of coordinates it is always possible. So, this is not a very stringent requirement as such, excellent.

So, using these notions of Class K and Class KR and so on. We now talk about definiteness of functions. So, before we go into definiteness of functions, if you remember just remind, reminder definiteness of matrices I hope all of you remember we sort of had a short discussion or whatever we stated a few things on symmetric matrices and then we talked about conditions under which a symmetric matrix is positive definite there were several equivalent conditions. So, A equals A transpose of course A in some n by n, so A is positive definite.

So, this is the notation if any of these conditions hold and any of these conditions hold first is that it is first that it is x transpose x strictly positive for all x in Rn, x not equal to the 0 vector the next one is that also, and write it in shorthand because you already done this next was that eigen values of A are strictly positive, all principle minors have positive determinant.

So, these are sort of the conditions under which a matrix is considered to be positive definite. So, positive determinants like I said so, this is I will just expand this all principle minors have positive determinant. So, I want you to keep this in mind when we talk about positive definiteness or definiteness of functions.

Now, we are talking about definiteness of functions. Excellent, excellent. So, what is the sort of the first such definition alright, we want to take a look at that. So, the first definition is that of a positive definite function. So, if we have a scalar valued function from R cross B r to R. So, what is this R cross B r and all that. So, I think we already know the notation. So, B r is basically the set of all x in Rn such that norm x is less than R. So, this is the ball of radius R around the origin practical notice that origin has been assumed to be our equilibrium and therefore, we sort of measure distances from the origin.

So, the ball of radius R is around the origin. So, we do not specify the center. So, what is this VR cross B R goes to R n and all that, before I even sort of go forward with the rest of the discussion. So, here R represents time, this represents the, a sort of ball around the origin of the states.

So, this V function takes time and states and gives me a real number. So, these are the very specific kinds of functions for which we are discussing definiteness we are not discussing definiteness for some arbitrary functions, we are discussing definiteness for very, very specific kind of function, which map time and which map states to some real value. So, what do we require, we of course require that this function is continuous, it is already scalar value is obvious, because I map to the reals.

Then we require that the function takes 0 value for all time, as long as the state is 0. So, if I plug in the state value to be 0, that is origin 0 means the origin of course, so if the state value is 0. That is if I am at the origin, then this scalar valued function definitely gives me 0 irrespective of what time I choose, irrespective of what time I choose.

The second important point is that there should exist a Class K function Phi such that this function V t x dominates this Class K function of Phi x. of norm x notice this notice the argument, norm is always non-negative remember, the argument of a Class K function also has to be non-negative, it definitely maps into non-negative values, but the argument itself has to be non-negative. So, this Class K function takes as its argument the norm of the states which we know to be non-negative always. So, Phi in there has to be exist such a function Phi such that the V t x dominates this function Phi of the norm of x for all t and for all x in B r.

So, notice, so, remember this function Phi is a Class K function in the norm of the states. So, the important thing is if these 2 conditions are satisfied along with continuity, then we say that the function V is positive definite, and it is denoted by V greater than 0, I mean just like we denote for matrices A greater than, just like we denote for matrices A greater than 0, we have a similar notation for functions also we say V is greater than C.

The other important point that I really want to highlight is that, notice that the Class K function the way we have defined, is it is continuous it is strictly increasing for all values of the argument and it is 0 and 0. So, it is continuous strictly increasing for all values of the argument and it is 0 at 0. This is important, but we need the function V to only dominate this function we do not particularly require V to be also strictly increasing.

This condition does not first of all does not imply V is strictly increasing, we just need to dominate a strictly increasing function. Also, we need the domination to happen only for a small ball around the origin, we do not need the function V to dominate this Class K function for all time.

So, let us look at a sort of picture and we will see what is the significance of this Br, Br is just some local domain around the origin. If you remember, our stability definitions also have the attractivity property, if you may, also has this Epsilon delta, where this delta could depend on initial time or not. But if you have initial conditions starting in this delta ball, then you converge to the origin, it does not say that you can start at any initial condition and go to the origin.

So, that was also local notion. So that local notion is being sort of going to be portrayed using this Br, ball Br. So, we do not need this function to dominate a Class K function for all states, definitely we want it to dominate for all time, the time thing is not flexible we need this domination to happen for all time, because notice that the right hand side does not contain any time argument at all, be the left hand side contains time, but the right hand side does not contain any argument.

Therefore, this has to hold for all time, no doubt about it, but it does not need to hold for all values of the state. So, if I again if I try to make this sort of a picture here I have x. So, this is sort of. So, suppose I have some kind of a Class K function. Let me make it very simple. Suppose I have like a linear Class K function, something very simple.

So, this is my Phi of norm x. In this case, x is a scalar. So, norm x absolute value of x, let us not worry about it, I will stick to the notation to be consistent here. Now, I want my V x to dominate this function, of course, notice that V at 0 is 0. So, obviously, I have to still start at 0, then I can do things like this I can do things like this. So, until if I may again. So, let me try to pick another color. So, until here, which is sort of say r, so until here until r.

I am just calling it V of x because I have not shown the time argument. I mean, I can even do that. But let us say this is V t of x, but the, but it becomes complicated. So, I am just going to call V x, I am going to say that there is no time argument involved in this particular case. Why am I saying that is because if I put the time argument, then I will have to draw another axes and show how the function also evolves with time. So, instead, I will just say it is a function of just the state x.

So, even if V is just a function of x and time argument does not appear at all and that is also allowed, by the way that is also allowed. See it happens. So, even in that case, I do not really need my V to be strictly increasing. I need it to dominate this Class K function only and that to not necessarily for all values of the state, but until a particular until the norm is less than r, I need the domination only until a particular r, and if this happens this function V x, so, they in this case V is said to be positive definite, in this case we said to be positive definite.

So, obviously, as we have mentioned Br is an open ball around origin and positive definiteness is related to the notion of asymptotic stability in the Lyapunov theorems positive definiteness is going to be directly connected to asymptotic stability. So, what would be simple examples? So, simple example would be say, I know that, I mean I will start to construct a simple example from the Class K function itself.

So, I know that 1 over norm x plus 1 is a Class K function. So, if they say x is in R 2, then I can use the 2 norm if this is x1 square plus x2 squared and in fact, I will use a squared here that is fine. So, this I know is a Class K function, this I know is a Class K function.

So, if I sort of sort of work backwards and if I say my V t x is something like just introducing some time argument here I can say something like e to the power t divided by norm x squared plus 1 and this it should be obvious to you that this is greater than equal to let us see, so, in fact, I can just simply take it as t, why make it more?

So, this greater than or equal to Phi norm x this is greater than or equal to Phi norm x. And so, this is obviously a V is positive definite function, because it dominates a Class K function. So, this is a valid example of Class k function sorry this is a valid example of a positive definite V function. So, anyway, so, I think this is we figured out some important things here.

So, what we have sort of seen today is few different function Classes, which are going to be critical in talking about definiteness. And from there we talked about the first notion of definiteness which is positive definite and we saw an example also of what is a positive definite function. So, we will continue again in this vein in the upcoming sessions and we will progress on slowly towards stating the the Lyapunov direct method or the Lyapunov stability theorems that is it, folks. Thank you.