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Nonlinear Adaptive Control
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Week 3
Lecture 17
Stability Analysis in Linear Systems

Hello everyone, welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control. I am Srikant Sukumar from Systems and Control, IIT Bombay. We are as usual in front of our very very cool representative image of this rover on Mars motivating us to study the design and analysis of algorithms that are going to be driving systems such as these.

So, until last time, what we were looking at was sort of a couple of examples illustrating the different notions of stability that we have seen. So, we had spoken about asymptotic, globally asymptotic, uniformly asymptotic, globally uniformly asymptotic, exponential and globally exponential stability. So, we had looked at several stability notions and I do hope that all of you are going to be able to remember all of these acronyms, which are sort of easier to remember than of course these full length long sort of words.

So, I do hope that all of you can remember these acronyms. The important thing to keep in mind is that we almost require always just to remember two properties. And these two properties are that of stability and attractivity and once we have a good handle on these two notions of stability and attractivity all the other definitions and notions sort of follow from there after that.

We looked at a couple of examples, this first one was an example of a system which was attractive but not stable. And then the second one was of course, that have a very, very standard pendulum dynamics written and state space form and this is actually a globally uniformly asymptotically stable system. So, this is GUAS and the Phase plane portrait of this system lot of sort of looks like a spiraling in curve in the x 1, x 2 plane, excellent.

So, this is where we were last time and what we want to do is sort of look at a few more examples, we want to look at a few more examples today. So, this is lecture 3.5. So, this is the fifth lecture of week 3. So, we continue our excursion into examples illustrating the different notions of stability.

So, this one is this x dot is minus sigma, x cubed with some positive sigma and in this particular case, I can because it is a scalar system. So, x is in reals of course, we can actually nicely compute the solution looks something like this. Of course, we are assuming that x t0 is some value x0.

So, with this initial condition description, the solution can be written in this very nice compact form. And once we have this, I mean the conclusion is of course, written here that it is globally asymptotically stable. Let us sort of look at it a little bit more carefully. The first question is that of stability, the first question is that of stability.

So, what can I see from here is that this solution can, of course be further maybe written in this form. Plus minus I still say retain this x0 divided by square root of twice sigma x0 squared t minus t0 plus 1. Something that we can do, we can write it in this form, or we can keep it in this original form, it does not matter.

It does not matter we can keep it in this form. Now, what do we want in a typical stability? Sort of for a typical stability condition? What do I usually require? I require that the absolute value, absolute value, which is essentially, I am going to keep my life a little bit simple. Absolute value of this is simply obtained by deleting the plus minus sign. And putting in absolute value symbol here if you may or simply writing it as 1 over square root of twice sigma t minus t naught plus 1 over x0 squared.

Now, what do we want, we usually want this to be less than epsilon, because 0 is the equilibrium. So, it is should be obvious that x is equal to 0. Even without doing any change of coordinates directly have my x e to be at 0 itself. Now, suppose I want this to happen. So, this is sort of what we want to happen and I want to find some delta, I want to find some delta.

So, what I would do is, of course, try to simplify this expression and so, on and so forth. So, what would it be? So, what I will do is that I will sort of first of all, I will make this bigger, so I can have more space to write. So, what I am going to say is that, I know for sure that let us see, let us see, what do I know for sure, I know for sure that 1 that I know. So, let me just say it up front, I know that as t becomes greater than t0, this is a positive number. As t becomes greater than t0, this is a positive number.

And so, this is the entire thing is of course positive. And therefore, this is adding some positive quantity and therefore it is reducing the inverse, the inverse is getting smaller, as t becomes greater than t0. So, the largest value that this entire quantity can take is when t is exactly equal to t0, because as long as as soon as t becomes greater than t0, this is going to contribute a positive quantity which is going to reduce this inverse. So, one thing that should be sort of evident to you immediately is that 1 over, we just write it like this.

So, that this is in fact, less than so, this is in fact, greater than so this is x0. Because what happens, so infact absolute value of x0, because let us see what happens, if I plug in. So, what did I say? I said that this quantity attains its largest value at t equal to t0, because as soon as t becomes greater than t0, the numerator becomes larger, and therefore the inverse become sorry, the denominator becomes larger and therefore the inverse is smaller.

So, largest value of this entire fraction is a t equal to t0. And if I plug t equal to t0, this guy goes away. And all I am left with this absolute value of x0. So, if I ensure that is the cool thing. So, if absolute value of x0 is less than epsilon then I am ensuring that 1 over square root of 2 sigma t minus t0, plus 1 over x0 square is also, less than epsilon because I will write it in completeness. This happens this is less than normal or absolute value of x0, that is less than epsilon.

So, what does it mean? It means that I can means that choose delta exactly equal to epsilon down and we are done. If I choose the delta exactly equal to epsilon, it means that my initial conditions will start within the epsilon ball. And if initial conditions start within the epsilon ball, I am guaranteed that the solutions, which are smaller than the initial condition are also within the epsilon ball. So, I can infact choose delta equal to epsilon. So, note that this is independent I mean, we did not even have to work for it, this is independent of t0.

So, what does it mean? It means that it means what it means this is uniformly stable. This is uniformly stable. Now, what can I say about attractivity? Well, I mean attractivity is rather straightforward, I mean, as I keep increasing t, here this is going to go to 0, it does not in fact, matter what initial condition I choose, not at all does not matter what initial condition I choose, it is just going to contribute here.

Sure, it is going to contribute here, It is going to contribute here, but irrespective of what is the size of this, irrespective of the size of this quantity this is definitely going to blow up to infinity and you are going to converge the solution is going to converge to the origin. So, in attractivity, it is in fact obvious that it is globally uniformly attractive. It is globally attractive sure. And the fact that it is uniformly attractive is also, evident by the fact that you have a t minus t0, it does not matter.

So, this this bound on this x0 is first of all global in fact, So, wherever it is global uniformity does not have to be thought of because delta is obviously all of Rn, all of R in this case, therefore, it is obviously uniform also.

Because the whole idea of being uniform is that the bound delta depends on t0, but if the bound is all of R, then it depends on nothing. So, obviously, it is globally uniformly attractive now if I combine these 2 properties, global uniform attractive and uniformly stable then what do I get, I get globally uniformly asymptotically stable.

Not just globally asymptotically stable, but it is globally uniformly asymptotically stable. So, this is a rather nice example because we could actually solve it first of all and of course, it has a rather good set of properties it is globally uniformly asymptotically stable. So, the origin is globally uniformly asymptotically stable for this system. Excellent, excellent.

So, Let us look at the next example. So, we are looking at a bunch of examples to get a good feel for what stable system, what asymptotically stable systems what I mean and so, on, what are attractive system what do these look like. So, there is a sort of a second order system x1 dot is x2 and x2 dot is minus x 1 minus 2 over 1 plus t x2. So, this is a sort of second order system.

Now, if you look at the solution, the solutions look something like this. So, this is of course I mean, let me be clear again. Let us see, I think the solution is created in this case, assuming that x1 0 is x1 0 and x2 0 is x2 0 like this kind of a notation. And if you have these initial conditions, these prescribed initial conditions you have this sort of solution by this is what your solution looks like.

I have to sort of look at this example again, I will verify this later on, but this sort of a system is non-uniformly asymptotically stable, it should be evident to you that it is definitely converging, because again 1 plus t is going to go to infinity as t goes to infinity, and therefore, all these terms are going to go to 0. So, attractivity is definitely rather easy. And it also, has stability.

We are not really proving it here, because it is not going to be very easy to prove in this particular case but you can sort of look at the phase plane portraits try to draw the phase plane portraits. And you will find that this is non-uniformly asymptotically. So, it is in fact stable but not uniformly stable. So, you do not have uniform asymptotic stability.

So, this is another set special case and this happens usually when there is a time dependence here then this sort of tends to happen when there is a time dependence in the right hand side, you sort of get some kind of non uniformity in your asymptotic stability properties alright this is pretty standard alright, what about the next system this is the sort of really favorite linear scalar system example very, very simple.

So, this is sort of x in reals and x t0 is x0. So, this is the sort of the very simple scalar linear system example and the solution of this is of course, very well known to everybody and it it essentially looks like this it is x0 e to the power minus k t minus t0. And we of course, claim that this is exponentially stable because, if you simply look at your, what happens with your state just look at your definition like what is the definition for exponential stability. In fact, Let us look at global exponential stability and linear systems.So, rather straight forward.

So, global exponential is requires the existence of just some constants a, b such that your solution satisfies this exponential decay for all t, t0 greater than or equal to 0 and for all initial conditions and so, you can see that if I choose a equal to 1 and b equal to k, then satisfies global exponential stability condition, satisfies the global exponential stability condition. And so, this is of course, globally exponentially stable.

So, this is in general true for linear time invariant systems, you have some really nice properties of course, the first anyway the first property for general linear systems is that asymptotic stability is equivalent to stability plus the state transition matrix. So, in case folks, you all of you have forgotten the notation. So, suppose I have a system of the form x dot is A t x with some initial condition.

So, this is what is a linear system it is time varying, but it is a linear system then, in this case, the solutions are written as using a state transition matrix as Phi t comma t0, x0, where phi is of course, the state called the state transition matrix. So, all of you are expected to have seen this if you are not please revise this terminology and notation for linear systems.

So, all linear systems solutions can be written in this form where this is a state transition matrix obviously, again it should be evident to you that Phi t, t0 belongs to n by n. So, if where x belongs to Rn. So, this is an n by n matrix, it is an n by n matrix map, which maps initial conditions to the current value of the state. So, as you plug in different t here, you get the state at that particular value of time this is what is our linear system solutions are written.

Hence it is this so, for asymptotic stability, notice that we require stability and attractivity. And in this case, attractivity is denoted by this and stability is just written as stability here, but in fact, there is an equivalent characterization of this also, which is some written here for linear system stability is actually something like this.

So, this is actually what it means, sorry, Let us go here. So, this is actually equivalent to having absolute value or norm of Phi t, t0 less than k t0 and limit as t goes to infinity by Phi t, t0 is 0. So, this is the stability part and this is the attractivity part, this is stability part and this is the attractivity part. So, these are sort of cool, there are simpler conditions, if you may, I mean, if you want to call them simpler conditions. So, this is actually LTI in this case. So, slightly simpler conditions for linear systems. So, stability is not just a generic epsilon delta sort of condition but and neither is attractivity, but it is sort of codified in terms of the state transition matrix.

So, all these properties can be sort of codified in terms of the state transition matrix. So, the other thing we say is for LTI systems, that is linear time invariant system, so, what is the linear time invariant system that is, so, LTI I am going to make this bigger again. So, I can write, so, LTI system is just x dot equals A x and here we do not even care about writing an initial time well fine, I will write it just for the sake of it and in this case, the solutions x of t are written can be written as x of A t minus t0 times x0, where of course, this is the exponential of a matrix this is the exponential of this matrix.

I hope all of you know what is the matrix exponential. So, in this case, if I get uniform asymptotic stability, if I get uniform asymptotic stability, so, of course, I have 2 things I have stability. So, I need a couple of properties in this case that you know that UAS in this case is equivalent to, because this is of course, the state transition matrix. So, this is equal into e to the power A t minus t0 less than some k t0 and also limit as t goes to infinity e to the power A t minus t0 again the norm is equal to 0.

And if this is it is not difficult to show, I mean, this is not really linear systems intensive course as such, but it is not difficult to show that and this is something you should know also, from your typical frequency domain knowledge that this can happen if and only if real parts of all eigenvalues of A are strictly negative and so, solutions all solutions are exponential decay.

So, this is I mean there is a lot of I mean linear systems theory in this that comes in if you are so interested I can even tell you about it. So, any matrix A can be of course written in its Jordan form. So, real Jordan form if you may p lambda p inverse, where lambda is of course, the Jordan, Jordan block called the Jordan block, and so, e to the power A t minus t0 can actually be written as p e to the power gamma t minus t0 p inverse.

And if real lambda is less than 0, this is equivalent to real lambda gamma less than 0 because Eigen values do not change until this sort of a similarity transformation again something that you should know from linear systems theory you have something like this. So, all your solutions actually look like xt is P e gamma t minus t0, p inverse, x0.

If you actually redenote your states as Z equal to p inverse x, then the solution is Z t is exactly e to the power gamma t minus t0, Z0 and for certain I mean just to keep the discussion simple that we know that for certain cases gamma is a diagonal matrix is just the Eigen value matrix I mean just to think of it simply there are no complex eigenvalues this just is a diagonal matrix containing the eigenvalues it is just a diagonal matrix containing the Eigen values therefore the exponential. So, diagonalizable cases what happens your Z t, looks like e to the power minus lambda 1 t minus t0, e to the power minus lambda n t minus t0, Z0.

For the diagonizable cases you have something like this and even for the non diagonizable cases only the real part matters. So, diagonalizable cases of course, the real. So, for the non non diagonalizable cases also, only the real part matters now, the complex part really just contributes to sinusoids and all it does not change the magnitude of the solution. So, Z is of course, converging exponentially to 0 as you can see, because all of these lambdas are so, this is actually this is actually positive the way we have denoted it, is it is lambda 1, lambda n 0 here and 0 here.

So, we know that all the lambda i's are negative by our assumption and all of these are negative then of course, z is going to 0 exponentially. So, implies Z goes to 0 exponentially because you can look at the expression. So, implies of course, that by this transformation that x is going to 0 exponential because x is just that scaled by some constant matrix.

Therefore, we are done this is we just show we started with asymptotic stability and we took this sort of convergence properties, and from these convergence properties, I know that all the real parts of all eigen values have to be negative and then I can reduce A to its Jordan block form. And it is and then we know that, it is diagonalizable with this Eigen values lambda of which the real parts are anyway negative. So, therefore, Z is going to 0 and x is simply P times n.

Therefore, that is also, going to 0 exponentially because of this expression, these are all exponential decays. So, uniform asymptotic stability for LTI systems is actually just exponential stability, linear LTI systems cannot do anything but exponential.

So, what we saw today, there are a few more examples of these notions of stability. And in the end, we also, sort of tried to understand how these conditions simplify or become a little bit more specific for linear systems and linear time invariant systems where solutions can be written using the state transition matrix. So, we will continue further on this discussion of stability next time and we stop our discussion for today here. Thank you.