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Nonlinear Adaptive Control
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Week 3
Lecture 16
Stability Analysis with Examples Part 4

Hello everyone, welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control. I am Srikant Sukumar from Systems and Control, IIT Bombay.

We are of course, well into our third week and as we saw last time, we have covered several important notions which include stability, uniform stability and attractivity properties and we are of course, going to be starting to be able to look at very very special properties, that is property of stability and attractivity for dynamical systems in closed loop, for algorithms that are driving systems such as what you see in my background. So, without delaying any further let us look at the specifics of what we actually learned last time.

So, last time we actually started to, with the Van-der Pol oscillator and we just used the phase plane plots like you can see here for the, with the x on the, with x here and x dot here x and y here, whatever you want to name the state as. So, the Phase plane plot showed to us that all trajectories starting at origin start to head towards the limit cycle.

And similarly, all trajectories starting outside also start to head towards the limit cycle for certain values of Mu that of course, we did not see in any detail. That is not of interest right now. So, rather interesting, and we saw that it does not matter what the value of Mu is all it does is it changes the shape of this limit cycle. But, eventually your origin still remains unstable. So, which was rather interesting for us.

Now, after that, we since we had already seen the ideas of stability and uniform stability, we wanted to go to the other set of properties, which relate to somehow system convergence which is the sort of property that we had verified using the Barbalat's Lemma on the spring mass damper system a few weeks ago.

So, these properties are of course, were of course named attractivity in the context of systems theory, and we defined 3 different attractivity properties for the first one was Attractivity, the second Uniform attractivity and the third was Global uniform attractivity.

Now, we also pointed out of course, that there was no such thing as global stability, because we have stable and uniformly stable there is no notion of global stability. But with attractivity, we also have a notion of global attractivity. We also verified each of these for the Messera example. That is this particular example that we were working out, this particular example that we were working out, we sort of verified which property it satisfies, and we saw that it is at best globally asymptotically stable.

So, of course, after seeing attractivity, and global attractivity, and global uniform attractivity and so on, we also spoke of the most desirable set of properties, which are the ones that are here. So, all these are, in fact, the most desirable properties, the first one and why do we come to these at the end is because these are simply combination of the already proven properties of stability and attractivity. And therefore, we do not need to define each of them in an epsilon delta terminology, because we have already done that for attractivity and stability.

So, if you see the first one was asymptotic stability, and I use the acronyms rather frequently, so I actually wrote them out for you at the left side, left hand side of this, so asymptotically stable or AS requires stability plus attractivity. Then we had I actually put in something in the middle here. So, because the Massera system was of course, more than an asymptotic stable.

So, therefore, in order to sort of understand what is the best property that Massera example satisfies, we also introduced the Global Asymptotic Stability definition, which is just stability and global attractivity and this is called GAS. Then you have Uniform Asymptotic Stability which is uniform stability along with uniform attractivity, we of course, verify that the classical Massera example is not uniformly asymptotically stable.

I would rather go to this first that is GUAS which is globally uniformly asymptotically stable. So, this should be behind after this because we are looking at increasingly powerful properties and therefore, I would rather put it above this. So, what is globally uniformly asymptotically stable it is uniformly stable as before, but then in the attractivity I add the global property, so, you have Global Uniform Attractive Stable, so, it is uniformly stable. And it is globally convergent, globally uniformly convergent.

So, you can start from any initial condition and you will reach the equilibrium which was assumed to be the origin for all of this. So, this is called Globally Uniformly Asymptotically Stable and so, GUAS. So, one of the things that I usually point out at this time is that, for all of these cases, you there is no specified speed of convergence, for example, something like a t0 over t, if it is so turns out, that your function x t is something like a t0 over t, then also you have global uniform asymptotic stability, this is not going to be too difficult to verify.

You can in fact, do that. I mean, I mean, let me be actually more precise, say this is I mean, t 0, over t times x0. Suppose this turns out to be the solution. I am not specifying any differential equation or anything, but suppose I tell you, this is the solution. So, this is also globally uniformly asymptotically stable.

Why is that? Stability is obvious, because it is sort of decreasing. In fact, let me say t plus 1, just do, make our life easy where I say t0 is greater than equal to 0. So, for this kind of a system, you can see that as time progresses, I am going to get smaller and smaller value of the states and therefore, as time goes to infinity, of course, it is attractive, it does not matter what initial size of the x0 ball was taken, I am guaranteed to converge for all possible initial x0 balls.

Therefore, this is globally attractive uniformity is of course not going to be a problem in this case either. So, that is something actually that is something I will have to verify. Let me actually make this now, let me make this something like this to be sure that it is also uniform. Suppose the solution is t minus t0.

Suppose my solution is this and just so that I can be sure of uniformity too. So, it is not difficult to verify that this is also going to be uniformly stable and also globally uniformly attractive. So, for this system that this is globally uniformly asymptotically stable. So, this is also this is definitely globally uniformly asymptotically stable. But, now if I choose another system, which is something like this, this is also globally uniformly asymptotically stable because it is not, it is still converging.

The global stability property is definitely not lost and uniformity is also not lost. I can promise you any any system whose solution has t minus t0 in it, is definitely going to be uniform. This is a small tid bit for you? Because you can if if anybody, if you have seen linear systems solutions, they always have t minus t0 as as part of their solution and never t and t0 separately for all time invariant systems so, great.

So, if you have a solution of either of these kinds, these are both globally uniformly asymptotically stable. However, one thing should be evident to you is that if you look at this guy, this is, this denominator is going to infinity much faster than this guy, why because of this squared term it is a quadratic. So, this guy is going to infinity much faster than this denominator and therefore, this is going to 0 much faster than this guy.

So, this is one of the sort of, if you may, drawbacks, if you may want to call it although we may not be concerned with it so much of the asymptotic stability definitions, whenever we say asymptotically stable, we never really talk about how fast we are going to 0, how fast we are going to the equilibrium origin in this case.

And this is what exponential stability sort of formalizes and this is of course, again, for one particular kind of rate, it is not necessarily all kinds of convergence rates. Here, we are always exponential stability, as the term implies always means exponential rate of convergence and same with global exponential stability.

So, it is exponential rate of convergence. So, it is not exactly codifying all possible rates, but because we are giving nice Lyapunov conditions later on. So, we care about, we sort of talk about these additional two definitions, which allow us to codify some measure of how fast their solutions are, in fact, going to the origin and we are not just saying that the solutions will go to the origin, they will remain stable and all that we in fact, saying how fast they are going to go to the origin.

So, what is exponential stability? It is rather simple, it says that if there exists constants, r, a and b, which are positive, now note nothing depends on time and initial time and all that. So, exponential stability, the way it is defined is naturally uniform. And hence, it is a very strong notion.

So, exponential stability requires the existence of these r, a, b constants. Such, that your state norm always remains within the a norm 0 e minus b t minus t0. And this has to hold for all t, t0 greater than equal to 0 and x0 less than r, I would rather make this a little bit more precise and I would say this is actually t greater than equal to t0 greater than equal to 0 and for all x0 in an r ball.

So, for all local that is a bounded x0 in some sense some ball of radius r. And for all initial time greater than equal to 0 and t greater than equal to the initial time of course, you have this kind of an exponential decay. So, you can see this right hand side is exponentially decaying because b is a positive quantity. And t minus t0 is going to increase as t goes to infinity.

So, there is an exponential decay to 0 the right hand side and this will happen if your initial condition lie within in a ball. So, you are having an exponentially decaying envelope or exponentially decaying shrinking ball, in which the state trajectories have to lie and this is called exponential stability. It has been of course, given the acronym ES.

So, one thing that is sort of well known and is something you will need to prove it, you should prove is that exponential stability implies uniform asymptotic stability, exponential stability implies uniform asymptotic stability. So, this is stronger than uniform asymptotic stability. The next one is of course the global version of it and what is the global version entail it just removes the initial condition ball as you would expect whenever we make anything global.

So, one constant actually drops off. So, it has only two constants now only two constants and it you have the same condition in fact, this is exactly the same and the initial conditions are allowed to lie within any anywhere in Rn, anywhere in Rn is allowed for your initial conditions. So, this is the only difference as this is always the case and whenever we are talking about global properties, all we do is we remove the bound on the initial conditions. So, this one of course, has an acronym of GES global exponential stability.

And of course, it is true that GES global exponential stability implies GUAS that is, global uniform asymptotic stability. So, again, this is something that you will have to prove that global exponential stability and global implies global uniform asymptotic stability.

So, this is a rather nice and strong property, if you can indeed conclude this for your nonlinear dynamical system. It is not very, it is very unusual, I might add to have exponential stability for a nonlinear system, for linear systems, this is an obvious thing. Again, for those of you who have seen solutions for linear systems, this is an obvious thing that linear systems always have exponential trajectories, these are exponentially blowing up or exponentially falling down.

And so it is not a big deal in those cases. But for nonlinear system, this is a rather rare property if I may. So, what we want to do is, we, of course, want to see a few interesting examples. We want to see a few interesting examples. The first I mean, of system which satisfy some of these properties or not, and so on and so forth. And so, the first one, is this rather complicated looking peice.

Of course, it should be evident to you that there is no way I can solve this kind. So, you are no longer looking at these vary scalar type systems which we can solve and then conclude and find Delta given an epsilon and so on, because as I already mentioned, those methods have rather limited applicability. Because beyond the point you cannot expect to solve a nonlinear system. And therefore, we start to look at these more complicated examples.

So, let us see what we have. Let us see what we have. So, for this complicated system, I am not even going to attempt to solve it or anything I will try to show you what the system trajectories look like. So, basically the system trajectories look something like this here. This is the x axis there is something like so this did not work. So, this is something like a bifurcation here, we are going to make this bigger sorry, I apologize. I am going to try this again.

Not doing very well with the picture of course. Let me try to make a trick. So, that is something like a leaf like bifurcation here. And the system trajectories outside of this sort of do this it is not the complete picture and the system so it is still going to 0 as you can I mean this picture I hope it is indicating to you that it is going to 0, so 0, 0 is the equilibrium it is not difficult to verify and the system trajectories inside do something slightly different I see this happened alright the system trajectories inside just form similar small petals, just form similar small petals.

So, these are like the state space trajectories this is the state space trajectory. So, I am plotting x1 on this axis and x2 on this axis. So, all trajectories outside do this come around and go to the origin, all trajectories inside do this. So, now, it is obvious that this system is now even just by looking at just this we did this for the Van-der Pol oscillator, we can do it for this sort of a system also.

So, it should be obvious to you that the system is attractive, write this in blue but it is not stable. Again, why is it not stable? So, I mean, again, you can suppose I do this by making some epsilon ball, I make an arbitrary epsilon ball around the equilibrium which is origin in this case.

So, let me make an arbitrary epsilon ball so, I made this epsilon ball as you can see. I am trying to center it, of course. I think it was more centered earlier than it is now. I think this is centered enough with the drawing skills that I have, so excellent. So, this is sort of centered at the origin this is like I said, this is an epsilon ball.

Now, what is the problem? I think all of you can see the problem, you can all see the problem from the trajectories I have drawn, it does not matter where I start suppose I start outside I will get out like this from this petal and get in here suppose I start inside the petal, then I still go out and then come back. So, this is certainly not stable.

Now remember, again, again and again, I go back to the stability definition, because usually there is a lot of confusion on that remember, I have to be able to find a delta for every epsilon the user gives me. A lot of students would come to me and say, but sir, I can always you know, draw this epsilon ball this very, very large epsilon ball and if I dropped this large epsilon ball sir, I can always, get all these trajectories to converge to the, or remain inside the epsilon ball or you can find definitely some deltas, which will sort of remain inside the epsilon ball.

But that is not enough, you need to be able to find me a delta ball for every epsilon ball I give you. So, this yellow thing that is out here is not the epsilon ball that I would give you, I would give you this really small tiny green psilon ball and then of course, you will have some trouble. You will not be able to find me a delta because everything that you give me every everything that starts inside this epsilon ball, forget a smaller delta ball, everything that starts inside the epsilon ball will definitely exit epsilon ball 1 way or another.

So, although we have the rather desirable property of attractivity that is you have nice convergence. But what is happening is that the system is going really far out to come back and converge, which is not allowed in stability, you cannot go really far out and then come back, then the system is not stable anymore. Sure it is attractive. And if that is all the property you care about, then well and good, but it is not a stable system.

And in a lot of circumstances, stability is a key property. Stability is a key property, if I have an epsilon ball and I want to remain in it, I should be able to find corresponding initial condition balls, otherwise they do not have. Stability actually gives you for those of you have experience of this frequency domain sort of terminology. The stability property actually tells you something about the transient behaviour. Because it is, because it is saying that if I am given an epsilon ball, then then I can always find a delta ball so that my trajectory is always remain within the epsilon ball always.

So, that is telling me something about the transient behavior, the bounds on the transient behavior. And the exponential stability or whatever, or the convergence and attractivity property that we have, that we have spoken about is telling me something about the steady state behavior and all of that even in linear systems theory, it is rather critical to have a handle on the transient behavior and the steady state behavior both.

It is not enough to have one or the other. In fact, a lot of people are happy to compromise on the steady state behavior they can deal with a little bit of accuracy in the steady state behavior that is even if your trajectories do not exactly go to the origin or the equilibrium they are okay but, then definitely care a lot about the transient behavior, you cannot have transients which are simply shooting up to dangerous values or absolutely bad, big values and then coming back to the origin as t goes to infinity.

So, in fact, most practitioners would reject any controller, which makes a system jump to large values before coming to the origin and they would basically stop the controller right when it starts to jump, so, this is absolutely unacceptable. So, both stability and attractivity together are key, and therefore the reason for us defining all these new properties here, excellent. So, this system is attractive and not stable. Let us keep that in mind. That is what it says here.

Let us look at quickly the dynamics of a pendulum. And we we have already seen the pendulum you physically seen pendulum. I mean, like even the grandfather clock that you typically have at your home, although that one is never stopping. But if you remove the battery, then it does this whatever I mean oscillates. Then it settles down, this is the standard pendulum.

So, that is the pendulum with damper that is and the dynamics of that with of course, normalized mass and things like that is something like x1 dot x2, x2 dot is minus sine x1 minus kx 2 this the Phase plane portrait of this system, again, this is not easy to solve, you cannot easily get solutions for a simple system such as this either, but what we will what you will simply try to draw are some Phase plane portrait.

So, what you have, this is again between x1 and x2 and your trajectories is wherever you start they look like spirals that are falling in and they look like spirals that are falling in. So, this is, I mean it goes on for infinite time and therefore, I mean this is both attractive. This is attractive in fact uniformly attractive, uniformly stable and in fact, globally uniformly attractive, and uniformly stable.

It does not matter where I start, I just start with a big spiral, but I will still go to the origin. And so, this is in fact, globally, uniformly asymptotically stable. So, what I say here is not complete, this is, in fact, globally, uniformly asymptotically stable system any way if you remember, we had sort of spoken of such cases when time does not appear in the vector field on the right, you get uniformity for free. It is globally uniformly asymptotically stable system.

So, what we saw today was the rest of the stability properties, whatever was remaining, we also saw that in order to specify a rate of convergence to the equilibrium, we also can define exponential stability and global exponential stability properties, which are very strong. And we worked out a few examples, like we saw a rather interesting example of a stable and attractive but non stable system. And then, we saw the standard pendulum example, which is both attractive and stable.

So in fact, globally, uniformly asymptotically stable. And so we have, we also sort of learned that both transient and steady state behavior are critical and transient behavior translates to stability, while steady state behavior translates to attractivity properties, excellent. So, we will do some more of this in the next session. Thank you.