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Nonlinear Adaptive Control
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Week – 3
Lecture 15
Stability Analysis with Example - Part 3

Hello all, welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control. I am Srikant Sukumar from Systems and Control, IIT Bombay. We are now well into the third week of our course on nonlinear adaptive control. And we are already well underway into learning the concepts that will help us analyze autonomous algorithms that drive systems such as what we see in our background. So, without delay any further let me sort of recap for us what we were looking at until last.

So, last time, we sort of tried to complete this example on of a system which is a classical example where Massera and which is solved in the book by Vidyasagar. And for this particular system, we show that it is stable, but it is not uniformly stable. So, this is a rather rather interesting sort of system. Rather interesting sort of system. And we did a little bit of interest, nice bit of work and analysis in order to try to see what kind of choice of delta must one make in order to remove the dependence on the initial time.

And what we concluded, this is sort of the highlight of last time, what we concluded was that we require, we are required to choose the smallest possible delta out of all the deltas that we get for different initial times. And that is what we need to do in order to make delta independent of t_0 and obtain uniform stability.

We also sort of looked at this sort of very, very briefly looked at what is this Van-der Pol Oscillator. So, what we want to do is of course complete discussing this Van-der Pol Oscillator system in a little bit more detail today before we go ahead.

So, what is this Van-der Pol Oscillator? We already said last time that this is a very, very, sort of classical nonlinear oscillator system, it is also immensely popular, because, we want because, there are certain scenarios in which you want to construct systems which have an oscillatory model. For example, pacemakers, that are used in parts of individuals who have sort of arthemia and other inherent issues.

So, what we want to do here is to look at whether it is stable or uniformly stable or unstable. So, this is what we want to do. And the Van-der Pol Oscillator system is given by this equation 1.6. And here, we have the handle of choosing this particular μ , we can choose this constant μ and sort of defines or changes the behavior of the Van-der Pol Oscillator.

So, the first thing we of course, do is write this 1.6 in the standard state space form. And for this we of course, choose states x and y as always so this equation 1.7 is what we will analyze the first thing to observe is that $0, 0$ is an isolated equilibrium. So, this is an isolated equilibrium point for this system.

Once we do that, if you remember what I had suggested was it we just make phase plane plots, we are not going to really try to solve this because it is rather difficult to write a closed form solution for this system. So what we do is we simply make a phase plane portrait of the system and this is what is shown in one and two figures one and two.

So, although it says for every μ here, it is plotted for particular value of μ the phase plane portrait and then it so happens that for any initial condition, your system actually converges to this sort of a limit cycle, this thing that you see here this is the oscillatory behavior of the system if you may.

For any oscillator, the behavior is always such that it follows some sort of periodic limit cycle at the end, there is a periodic limit set and this is called the limit cycle. So, so, we already know, well we do not, we have not already defined it yet, but anyway, so, this is called the limit cycle behavior, because it is a cyclical behavior in the phase plane.

So, this is what happens wherever you start it does not matter where you start, you will always end up into this following this sort of limit cycle and this is true for most values of μ it is true for most values of μ .

So, so, now, if I want to talk about stability of the origin, what do I require? I require that given any ϵ positive, I must be able to find the δ such that if initial conditions are within δ , then my solution lies within an ϵ ball of the equilibrium. Now, if you now, we stated very carefully here we say that if you choose an ϵ which is smaller than the sort of the radius of the limit cycle and this is given by an example here, so, the limit cycle is this A , set A which looks something like this.

And if we choose an ϵ ball which is this guy, you can see that every solution is eventually converging to this cycle A .

Therefore, it does not matter how small a δ I make out here my solution is definitely going to get out of this ϵ ball and converge to the set A and because of this sort of unusual behavior. The equilibrium $(0, 0)$ is in fact unstable, because I can choose any ϵ ball smaller than this it has to lie inside the limit cycle of course, so, if I choose any ϵ which lies inside the limit cycle, there is no possibility of finding a δ because for every initial condition inside this ball, you will always go to the set A , and this is a problem.

In terms of mean it is a problem not in general, but it is a problem in terms of how we have defined stability and how we have defined stability. So, for stability we require that if the user gives me any ϵ or it can be anything, arbitrarily large, arbitrarily small, it does not matter for every possible ϵ that the user gives me, I must be able to give or provide a δ says that initial conditions inside the δ ball will remain inside the ϵ .

Now, in this case, this is absolutely not possible, as you can see, and because anything inside this ball will always go to A and we know that δ cannot be larger than ϵ . That is obvious.

So, equilibrium $0, 0$ is in fact unstable, however this is a very nice system. I mean, there is a very nice and well behaved system in the sense that all trajectories seem to go to the limit cycle in fact trajectories for certain values of μ trajectories outside also go to the limit. For certain values of μ the trajectories outside the limit cycle also converge to the limit cycle and for all values of μ trajectories in starting inside the limit cycle will converge to the limit cycle.

So, this is a rather nicely behaved system, but it is not stable in the sense of. So, remember that although we have painstakingly defined these notions of stability in the sense of Lyapunov and all, but it is not final holy grail, I mean there are systems which behave nicely, which do not exhibit stability in the sense of Lyapunov, this is rather nice.

Of course, this this might also lead us to thinking about more general definitions of stability, it is my, but then of course, we do not look at that in this course. But yes, it does make sense to think about more general definitions of stability, etcetera, great, great.

So, now that we have sort of have a decent handle on what is stability and uniform stability, we want to look at more additional properties that the dynamical system might possess. And, and we have already sort of seen in this Barbalat's lemma analysis that we are very interested in convergence, and we always want things to converge to 0 , functions to converge to 0 . So, this is the property that we are rather keenly interested in.

So, now this convergence property is formulated in systems theoretical language as attractivity. So, this is the notion of convergence for dynamical systems. So, suppose, in order to talk about attractivity, what I do is I assume that my equilibrium is in fact, at the origin it is not such a difficult assumption to satisfy if this does not happen, I simply do a change of coordinates.

That just is and once I do this change of coordinates, it should be, it will become evident to you that this equilibrium in this y coordinates is in fact, the origin, this is just to make our notation easy. That is, and it is very standard practice. So it is not something crazy, it is just very, very standard, it is not even a big deal, we always consider origin to be our equilibrium.

So, what is what does it mean for for a origin to be attractive, it means that for all initial times, there exists a δ depending on this initial time only the initial time such that initial conditions within the δ ball result in all state trajectories converging to 0 as t goes to infinity. So, this is essentially convergence, essentially convergence the way we know the only thing we have done is we have fixed some initial time and initial states.

So, for attractivity I require I am required to find the initial condition ball of size δ , this δ is of course allowed to depend on t_0 , unlike the stability case, here there is no ϵ . So, there is no ϵ dependence. But t_0 is of course there so it can depend on t_0 . And we required that if the initial conditions are not within this δ ball that we have provided, then all my solutions, my solution, the state trajectories, in fact converge to 0 as t goes to infinity.

As usual, we also have the notion of uniform attractivity. So, what is uniform attractivity? It is that for all t_0 , there exists a δ now this δ does not depend on t_0 either. Therefore, I mean,

so let me be a little bit more precise and here δ positive, similarly δ positive, this is important, let us not miss this. So we need the δ to be of course a positive quantity.

Now here, the δ cannot depend on t_0 either, so it is just a constant for uniform attractivity. You need to be able to find a positive constant δ such that initial conditions within the δ ball guarantee that your state trajectories converge to the origin as t goes to infinity. So as before, as in the case of the definition of stability, versus uniform stability, here too the uniformity is with respect to the initial time, that is what we mean by uniformity in a system theoretic setting. So, I hope that is clear.

So, now that we have these two new notions, why do not we test these for the example by Massera that we have already seen. So, if you look at what I am going to do is for the Massera system, I am going to write this solution out again. So, I will write out the solution again.

So let us see if I can in fact do that. I can, if I can copy it. So it does allow me to copy. I hope it let me paste it to, so here you go. So, so this, so this was I guess this was our example. I mean, this was this was the example for Massera, example. So this is what is our was our solution. And, what do we want to do? We want to sort of understand whether this is attracted, and slash all uniformly attracted.

So, let us see, is this an attractive system. So in this case, I just need to find a δ . So, I just need to find a δ , like so find δ , for convergence. So, that is all I need to do. I need to find a δ for convergence. Now the question is, what sort of initial conditions are allowed? So, let us notice something, let us notice something.

So, once I fixed this t_0 . So in fact, it does not matter whether if it is t_0 or not, but, but let us notice something, this is this is some value here with some, constant, like. Similarly, γ is also some constant. After t_0 x_0 is fixed, for a particular choice of t_0 and x_0 , it does not matter what these choices are, in fact, but once I have chosen this t_0 and x_0 , these two are in fact constant quantities or they are not changing with t . Notice they are not functions of t . So, there is no t dependence in either of these, t appears only here.

And what do I know about this stuff, I know that this is going to be exponentially decaying going to be exponentially decaying. So, this guy once t_0 and x_0 fixed, for a particular t_0 and x_0 , these two are fixed quantities, this guy is definitely exponentially decaying. So, as t goes to infinity, this is going to 0, rather fast. So, an exponential decay.

And therefore, it does not matter what these quantities are. And the important thing is they do not change with t . And because they do not change with t , this entire thing is going to go to 0 as t goes to infinity.

Therefore, I hope you believe me when I say that δ is infinity. So what I can say is for all x_0 in \mathbb{R}^n , I have $\lim_{t \rightarrow \infty} \text{Norm } x_t = 0$. So in fact, it is not just while it is definitely uniformly attractive, that is for sure. It is uniformly but it is also globally attractive. It is not just uniform, but also globally attractive.

And this is what. So this is from the previous example. This is what is our next definition, globally uniformly attractive, that is for all t_0 and x_0 , $\lim_{t \rightarrow \infty} \|x_t - 0\| = 0$, this is in fact, our next was our next definition, this was in fact, our next definition. So this is a rather strong property, rather strong property.

That this Massera system that we were considering, was globally uniformly attractive or uniformly globally attractive, however you want to say it. It was not just this, but something more because there is the delta can be pushed to infinity. So, it is a rather powerful property.

So this picture, I mean, I know we did not sort of discuss this, but this is an illustration of what happens when the Van-der Pol Oscillator just spend a minute on it before going further, for the Van-der Pol Oscillator, we had this phase plane for a particular μ . But for all μ 's this is the only difference that happens is as you change μ this limit cycles sort of seems to change in shape a little bit, otherwise all trajectories starting at origin still converge to the limit cycles.

So, the only thing that changes is the shape of this limit cycle. So, the origin continues to remain unstable as we stated before for the Van-der Pol Oscillator. So, this is, so, as it is mentioned here, the limit cycle begins as a circle and with varying μ becomes increasingly sharp at the corners here and here you see becomes sharp, as you increase them. So, the nature of the limit cycle changes. So, depending on your application, you may choose a particular μ , but the fact that the origin is unstable, continues to be as it is.

So, we have already seen stability uniform stability, and now we have seen three attractivity properties, there was no notion of notice that I want to say this note no such thing as global stability, as far as the stability definitions go, there is no such thing as global stability, we saw stability, we saw uniform stability, so two different definitions here because we are always requiring us you always are required to choose an epsilon and correspondingly get a delta. So, a delta cannot be greater than epsilon, we already know that.

So, therefore, it is not possible to say that delta can be arbitrary. Delta cannot be arbitrary. delta definitely has to be less than or equal to epsilon. So, delta spanning all of \mathbb{R}^n is not allowed. Therefore, there is no such notion as global stability. It does not make sense at all. However, in case of attractivity, we saw three different definitions, we saw attractivity, we saw uniform attractivity, and we also have global uniform attractivity that is what is this definition, that is, does not matter what your initial condition and initial time are, their trajectories do converge to 0 as t goes to infinity.

And we in fact, saw the Massera system that we have, is globally uniformly attractive. So, now we will continue to look at this Massera example and talk about each. Now, the rest of the definitions are rather straightforward. And the rest of the definitions are rather straightforward and incrementally strong.

So, let us look at asymptotic stability, what is asymptotic stability? It is simply combining stability and attractivity. So, so, we already know that for example, about the Massera system that is stable, we already know that it is attractive therefore Massera system is asymptotically stable. So, the Massera system is in fact asymptotically stable.

Let us go to the next one. The next one is uniform asymptotic stability. So, we have added the qualifier uniform. So, therefore, the qualifier uniform also goes into both of these definitions, that is it is uniform, least stable and uniformly attractive.

Now, the question is, is the Massera system uniformly stable and uniform? Is it uniform asymptotically stable? The answer is no. The answer is no. Why? Because, it is uniformly attractive. Sure, we just proved that it is uniformly attractive because there is no delta and delta is infinite.

But it is not uniformly stable. It is only stable. Therefore, the Massera system. So, this I will, I mean, I will start to make these acronyms asymptotic stability, the uniform asymptotic stability exponential stability, not, so the Massera system is not uniformly asymptotic stable is asymptotic stable, because it is at the origin of so whenever I say something is stable uniform, asymptotic is stable. And I do not mention the equilibrium. I am just trying to shorten the sentence.

Whenever you are talking about any of these notions, you are always talking about a particular equilibrium. So, the Massera system is asymptotically stable at the origin, the Massera system is not uniformly asymptotically stable at the origin.

So, when I write it out, formally, I have to write the whole sentence, please keep this in mind, you have to write out the whole sentence. I am not saying the whole thing occasionally. Because it is clear from the context that I am talking about the 0 equilibrium. So, the Massera system is not uniformly asymptotically stable, because it fails on one count and we need both of these.

Let us skip exponential stability and go to the because exponential stability is the strongest property in fact, so it should come in the end. So, this is the next property is global uniform asymptotic stability. And what does it require? It requires global uniform asymptotic stability. So, it requires uniform stability and global uniform attractive.

Now, it should be obvious to you that because the Massera system is not uniformly asymptotically stable, it cannot be globally uniformly asymptotically stable either. However, there is of course, a definition somewhere in between which is not really mentioned here, but I will still put it for reference.

Well, let me put it say here global asymptotic stability, so, this is G, so, this is, this is GAS. And, what does this require? It requires stability because there is no uniformity plus globally attractive, stability plus globally attractive. And you will note now that Massera system is globally asymptotically stable.

Because it is stable and it also is globally attractive. So, so, this is not the best property. So, so, whenever we have a system, we are always trying to find the best property that is satisfied out of all of this. So, asymptotic stability, yes, as Massera system is stable, but it is something more

because the system converges from any initial condition to the origin that is something stronger than just asymptotic stability, which is a local notion.

Therefore, this notion of global asymptotic stability also exists, which is that this Massera system also satisfies this, that is that the origin is stable and also globally attractive. So, the Massera system is in fact a globally asymptotically stable system.

I hope that is clear. So, so, we have done some interesting things today, I will sort of try to summarize what we have done today. So, we of course, started off with talking about the Van-der Pol Oscillator, we saw phase plane portrait to sort of judge whether it is stable, uniformly stable or unstable.

What we found was that for different values of μ , the limit cycle shape changes. But, overall, the stability question has the same answer, that is the origin is always unstable. It is neither stable nor uniformly stable. Then we went on to talk about additional definitions beyond stability and these were related to convergence and, so this is, these properties are called attractivity properties.

So, we talked about attractivity, we talked about uniform attractivity, global uniform attractivity, global attractivity, then we talked about asymptotic stability, global asymptotic stability, uniform asymptotic stability and global uniform asymptotic stability.

So, we will sort of do this again a little bit next time. But, what we were able to conclude for this Massera system, this example that we had been talking about the whole a lot is that it is the best property that is satisfies it globally asymptotically stable, that it is stable and it is globally attractive together. So, that is it folks, so we will meet again next time soon. Thank you.