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Nonlinear Adaptive Control
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Week – 3
Lecture 14
Stability Analysis with Examples - Part 2

Hello all welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control. I am Srikant Sukumar from Systems and Control, IIT Bombay. So, we are again starting in front of our motivational image of this rover on Mars and we are now well into studying mathematics which will help analyze the stability of algorithms driving systems such as this. So, without delaying any further let me first give a short recap of what we were doing last time.

So, last time we had started looking at the notion of uniform stability. So, this was the week 3, lecture 1, and here we saw sort of what is the difference between uniform stability and the notion of stability. So, essentially uniformity is here with respect to the initial time.

We then started to look at an example from the book of Vidyasagar, this is a very, very classical example by Massera and Vidyasagars book works it out, why this is such a nice example is because A it is a scalar system, which can be easily integrated to compute the solutions and B it sort of gives us some very nice features of stability versus asymptotic stability and things like that, so, this is rather nice for us.

So, what we had seen until last time, what we had completed until last time was the proof that this system is in fact stable in the sense of Lyapunov. Now, in order to prove the stability, we had given an epsilon computed a delta. So, this was the critical component, we computed a delta. So, we will continue to work with this delta, even today, so, that was what was required for us to prove stability given an epsilon we wanted to be able to compute the delta.

So, once we did that, we were able to compute stability. And the next endeavor for us was to talk about uniform stability. So, we started discussing it. But then we did not have enough time. And of course, we were missing a few steps. So, today, I am going to start there, and what is it what are we going to try to do?

We are going to try to prove uniform stability. And we know from the definitions here, that the only difference is the absence of dependence on t naught in delta. So, we need to be able to find a common delta for all possible initial times, that is the whole idea, and we need to find common delta for all possible initial times that is what is the uniformity.

So, now, we do see that, in our case, where this delta, this initial time dependence comes from and that is precisely from this gamma, everything else is in fact, independent of initial time. It is only gamma, which depends on the initial time and what is this gamma, this gamma was this quantity, this gamma was this quantity right here. It is just a quantity, which is a function of the initial time.

Now, what do we need, so, let us carefully think what we need. So, given any t0, I can find a gamma corresponding to this particular t0. And then using this gamma, I can compute the delta. Now, the question that arises is what to do if I want a common delta for all possible t0. Now, let us, so this is where we need to be careful.

So, let me do before I even try to prove for the specific case even if even before I try to prove anything for this specific case, let me try to give some kind of intuition for how to choose such a common gamma in the general is such a common delta in the general case. So, let us see.

So, suppose I have some way to make this a bit bigger suppose I have t0 1 and corresponding to it, I get a delta 0, sorry, I am going to call it a delta 1. Similarly, I get another initial time and corresponding to it I get to delta 2 and similarly, I go on and on I get some delta k, given k and I know that this is of course, some kind of an infinite series because my initial time can vary from whatever, whatever initial value say 0 to infinity.

So, I can have all sorts of initial times. Now, what does which delta should I choose is the question so, I have so many deltas here so, which ones should I choose, which one should I choose, here is the question. Now, let us look at the stability definition what does it say for all epsilon positive there has to exist some delta such that if initial condition is delta away then my solution is epsilon away.

So, notice that this entire I mean although I did not mentioned it but so, what although mentioned it explicitly all of these are corresponding to correspond to same epsilon. Suppose I use the same epsilon and I get different delta values corresponding to so, these are all different initial times if I may. Here notice that the subscript gives me dependence on initial time and this quantity here shows me dependence on the epsilon.

So, basically all of these are for the same epsilon, I want to indicate that here. So, once I have this list, the question is which one should I choose? How do I know which delta to choose and for that we need to refer to this definition here. That if my initial conditions are within delta then I want my state trajectories to be within epsilon.

Now, usually when you have an infinite collection like these like this then the problem then usually what are the choices for which one I should choose. The usual choice is the largest or the smallest of this, usually the smallest or the largest of this and because there is an infinite collection, I would say the supremum or the infimum of this sequence somehow, I will take the supremum or the infimum of the sequence.

Now, let us see what happens if I take the supremum. So, say delta equal to sup over k delta k, so k greater then equal to 1. So, suppose I take delta to be the supremum, then what happens? If I take delta to be the largest possible one and then what happens, then I am surely suppose I mean I can show you what sort of problem we landed.

So, suppose delta 2 less than delta, just suppose I mean, see because it is obvious that delta is the largest value of all of these, therefore, there must exist some k for which delta k is less than delta. So, I am just simply saying that let us say it is delta 2 and say delta 2 is less than delta and because it is after I choose the delta to be the largest one. So, obviously, there are some deltas which are less than this.

So, suppose delta 2 is smaller than delta, then what happens? So, what I know for sure, if I choose t0, 2 as my initial condition as my initial time, at t0 2 what happens?

I know that norm of x0 minus xe less than delta 2 implies norm of xt minus xe less than epsilon. I know this. Now, what is the problem? So, what is the problem? The problem is I have chosen the delta as larger than this. So, the question is it true that x0 minus xe less than delta implies xt minus xe less than epsilon, what do you folks think? So, this delta here is a bigger ball than delta 2.

I am guaranteeing that if I start in the small delta 2 ball, I remain my solutions remain in the epsilon ball. But now I am asking that if I start in the larger ball, will I still remain within epsilon?

So, the answer is obviously, that this is not guaranteed. If I start in the if at time t0 2. If I start in the smaller delta 2 ball, I am guaranteed to remain within the epsilon ball. But if I start in the larger delta ball, I am not guaranteed to remain within the epsilon ball. So, the choice of choose choice that we made of the largest delta k is definitely not a viable choice, definitely not a viable choice. So, what is the other choice available to us?

Suppose now, I choose my delta as the smallest one, suppose I choose my delta as the smallest one then what happens and again, so, I know that this delta is the smallest one. Now obviously, I cannot say that implies obviously that delta i greater than equal to delta for all i because I chose the smallest value. So, obviously, all other values have to be greater than or equal to this value.

So, for t0 i is arbitrary i, what can I say? I know that x0 minus xe less than delta i implies xt xe less than epsilon. Now, does this also imply that x0 minus xe less than delta implies xt minus xe less than epsilon. And the answer is an emphatic yes, why? Because delta here is smaller than this, so, if I start in a larger ball and my solutions remaining epsilon 1 it is guaranteed that if I start in a smaller ball, my solutions have to remain in that epsilon 1.

So, the picture would go something like this, so, let me try to make these so called smaller and larger balls, here we go so, this is the say the smallest and the largest ball and just trying to adjust the center, give me a moment. So, this is more or less the center. So, what am I saying? So, this guy and then I have this guy and then I have this guy.

So, this is of course, the epsilon ball, this small blue thing is the delta ball and then the small black thing sorry the small blue thing is the delta I ball and the small black thing is that delta ball. So, what is evident to me that if I know that my solution starting in the delta i ball remain in epsilon ball, so, if my solutions that start in the delta I ball remain in the epsilon ball then it is guaranteed that if I start in the delta i delta ball which is smaller than the delta I ball so, anything inside the delta ball is also inside the delta I ball. So, anything starting here also remains within the epsilon ball.

So, let us this is evident. So, this y even without this picture, I can still claim that this implies that norm x0 minus xe is less than delta i. So, what do I know? So, this is a valid choice of delta, so, what do I know, I must choose the smallest of the deltas possible. So, if I have a sequence of initial times and correspondingly I get a sequence of deltas, I must choose the smallest delta to get uniformity.

And notice that this of course is independent, I mean, it should be obvious to you that this is independent of initial time because I took infimum over k and k is what was our initial time dependency, k was the initial time dependence. So, that is the first thing I need to know that in order to remove the initial time dependence, I need to choose the smallest possible delta. And that is important. Now let us look at our specific case. Let us look at our specific case.

We have epsilon over gamma eM and the gamma is what depends on initial time. Now the idea is I need to choose the smallest of the deltas, so what is it?

Let me be careful. So, what do I want to do? So, I want to, let me be more formal. I want to find inf over t0, take greater than equal to 0 does not matter for t0 greater than equal to 0 of delta t0 epsilon. And this is same as if you may, I hope you find because what was delta, the way we founded the delta way we computed was epsilon over gamma eM, and gamma is was the only quantity that was bringing in the initial time dependence.

Now if my I want my smallest delta, I would need my largest gamma because gamma is in the denominator. So, there is an inverse relationship. So, therefore, this makes sense. I need to find my largest gamma.

So, then, if I move forward, what is the largest gamma? So let us see, sup over t0 greater than equal to 0, so let us look at the expression. So, it is I believe, let us copy it let us see if I can copy it, so, I make it smaller so this is what is my sup over, so I need to find the supremum of this quantity. Now, it should be obvious that exponential is not damaging the supremum. So, this is actually then I mean that is fine. So, I will essentially say that this is equal to exponent of the supremum over initial time of minus 6 sine t0 plus 6 t0 cosine t0 plus t0 squared.

Now, can some of you already guess where we are going with this? What is the largest possible value of this quantity for arbitrary t0? That is when we start to hit the problem. This is actually equal to infinity or I mean, if you may, if you want to be more precise, this is tending to infinity, as the t0 goes to infinity, so last time, we spoke about this in terms of increasing functions and all that was not correct, let me reiterate, it was not correct.

This function is not monotone or anything. But one thing is clear about this function is that the largest value that this function can take is infinity itself, this function blows up, why, the same argument is before. This t0 square is going to dominate all of these guys. It does not matter what happens to this, this is just between plus minus 6, this is going to be between plus or minus 6 t0.

But then this t0 squared, which is a quadratic term is definitely going to dominate all these guys. So, this entire quantity goes to infinity as t0 goes to infinity. So, what happens to the supremum, it means that the supremum itself goes to infinity as t goes to infinity, so what happens to the delta, implies delta which is inversely proportional goes to 0 as t goes to infinity. And this is not allowed. And we need the ball to have a nonzero size like this, this ball has to have a non-zero size. But here what are we getting? We are getting that our delta actually goes to 0 as t goes to infinity and this is certainly not allowed.

So, therefore it is not possible to choose a delta independent of t0 in this case. So, what does it mean? It means that the system we have is stable but not uniformly stable. So, now of course I mean there is an additional statement here but, but we will talk about this a little bit later because we have not defined asymptotic stability yet, we have not defined asymptotic stability yet. So, we will talk about this at a slightly later stage.

So, one of the very interesting systems is that the van-der pol oscillator is a very, very commonly used system to design pacemakers for the heart and other oscillatory dynamical systems and very, very interesting system and with the dynamics given by equation 1.6 and for different values of mu the behavior of this oscillators changes.

So, the question is what happens, what can you say about the stability of the origin for different values of mu, different values of mu, what can you say about the stability of the origin for the Vander pol oscillator. Of course, in state space form it can be written in this way, in the state space format it can be written in this equation 1.7. So, what I want you to do is to actually try this out.

So, remember one thing you will not be able to solve this equation for solutions just like we did for this very special problem it is very difficult to actually solve this equation. So, I do not recommend that, what I recommend is for you to make these phase plane plots like this, plot between the x and y that is the two states x1 and x2 or x and y whatever you want to call it. And I want you to see the how the picture looks, how the phase plane plot looks. And based on that I want you to comment whether the origin is a stable one or not.

So, I would like you to look at the vander pol oscillator just look at the phase plane plot not do any analysis, not try to find delta corresponding to epsilon and so on and so forth. But I want you to comment just by looking at the phase plane portrait which is the plot between x and y states.

So, this x and y states and comment on the stability of the origin whether it is stable uniformly stable or not. So, one of the important things to remember is that if xe in neither stable nor uniformly stable, then it is unstable this is the definition of an unstable equilibrium if it is neither stable nor uniformly stable, then it is unstable.

So, I think we saw some rather interesting concepts today. So, we were continuing the problem of trying to comment on the stability for a very particular classical system. And we already looked at stability and we were trying to see if we can also get uniform stability, it turned out that it was not possible and in order to remove the initial time dependence on the delta, which is what we required to prove uniform stability.

It so, turned out that this was not possible, because we had to choose the smallest delta and the smallest delta in this case was tending to 0, which is not allowed as per our stability definitions. Therefore, we could see that the system turned out to be stable but not uniformly stable at the equilibrium which was origin in this case. So, this is way, we stop today and let us meet again soon. Thank you.