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Nonlinear Adaptive Control
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Week – 3
Lecture 13
Stability Analysis with Examples - Part 1

Hello folks, welcome to yet another session of our NPTEL on Nonlinear Adaptive Control, I am Srikant Sukumar and from Systems and Control, IIT Bombay. We are as usual in front of this very nice motivating background image of this rover on Mars. And we expect that we are moving forward towards learning how to analyze and design algorithms that are going to drive autonomous systems such as these.

So, without any further delay, I am going to sort of try to recap what we have done until last time. So, at the end of the last week's lectures, we started looking at stability in the sense of Lyapunov.

And in the beginning of it, we sort of tried to look at what is the notion of existence of unique solutions, which is sort of a critical ingredient for us to be even able to talk about existence of stability and so on and so forth and other properties of any differential equation for that matter. Once we understood that, we required these Lipschitz notions in order to talk about stability and talk about existence and uniqueness, we went on to define the notion of an equilibrium and especially the notion of an isolated equilibrium. We try to understand why this isolated equilibrium is such a key concept in studying stability.

And the fact that non isolated equilibrium does not even allow us to talk about convergence and comparison with one specific equilibrium point because there are so many of them arbitrarily close to each other. Now, so, this is what, this is why this was a such an important notion.

Once we had under our belt the notion of an isolated equilibrium, we were ready to talk about the first definition which was stability in the sense of Lyapunov and this is in the format of a very, very standard format of an epsilon delta definition. So, this is the standard format for any stability definition that we will be seeing now or in the future. So, what is it, it sort of comes as a challenge question that is given every epsilon positive or even any epsilon positive, I should be able to generate a delta, which depends on possibly the initial time and the epsilon.

And this is also a positive number such that if my initial conditions start within a delta ball of the equilibrium, then my state remains within the delta ball of the equilibrium or remains within the epsilon ball of the equilibrium for all time, and we do this very nice.

Now, illustrative picture in order to understand this notion, we also understood that delta has to be less than or equal to epsilon, what thinks to make sense. So, once we had the notion of Lyapunov stability understood, we want to look at the notion of uniform stability. So, what is it? So, this is where we begin today.

So, what do we have in uniform stability the only difference between Lyapunov stability and uniform stability or uniform stability in the sense of Lyapunov versus stability is the word uniform of course, So, we added a new word new qualifier, if you may, so, that is of course, one new thing and what does it entail this uniformity?

This uniformity is with respect to the initial time. So, if you look at the deltas that you get, in each of these definitions, there is a difference that the delta in the case of uniform stability is independent of the initial time and so this is what is the difference everything else is exactly the same.

Given an epsilon, there has to exist a delta which now depends only on epsilon not on t naught such that if my initial conditions are within the Delta ball, my solutions remain within the epsilon ball of the equilibrium for all time.

So, this statement and this statement are absolutely the same, if you notice there is no difference between the two. The only difference lies in the delta wherein here it was dependent on t naught and here it was independent of t naught and we will of course see when such a situation arises, we will look at examples subsequently of when we have one property and not the other.

Now, one of the important things to remember is that we say that the equilibrium is unstable. If it is neither stable nor uniformly stable, if the equilibrium xe does not have either of these two properties, then we say that our equilibrium is unstable, then we say that our equilibrium is unstable.

So, in order to sort of illustrate these properties, let us look at an example, the very, very standard textbook example, from the book by Vidyasagar. This is from the Vidyasagar book, you can find it in this book of nonlinear systems by Vidyasagar. This is in the chapter where you know stability and other notions are in fact being defined, a very, very, very standard example, in that sense.

So, what is it, it is a very simple scalar system, you are given x as a scalar quantity, so I will sort of explain, sort of explain x is in our real number. And you have some initial condition, of course, x, t0, is just your initial condition I am not denoting. It is not denoted here as anything else, but x, t0. So, it is x dot is 6t sine t minus twice t times x, so and of course, like I said, it is a real number with some initial condition at initial time.

Now, if I, once I have chosen an initial time, and this initial state, of course, I can integrate this, this is very easy to integrate, because this is simply dx over x, and this is 6t sine t, minus twice t dt. And I can of course, integrate both sides. So, from x t0 to xt on the left, and t0 to t on the right, so, this is not very difficult to do. I can actually carry out this integration is this expression; and the outcome of this integration is exactly this expression, so the outcome of this integration is exactly this expression. Let us not be very worried. So, the left hand side actually gave me log x, so basically, from this guy, I got log x, log x divided by x t0.

Therefore, I got an exponential here, because I had log x divided by x t0, and therefore I got an exponential here, when I took it to the other side. And the integral of this was simply 6 sine t minus 6t cosine t minus t squared.

And this is the responding to the initial time, this is corresponding to the upper limit t. And this is corresponding to the lower bound t naught, very standard in definite integral. So, now, it is a rather long expression, but not too complicated. You can actually take a derivative with respect to time of this guy. And you can see that it satisfies your differential equation, it is not difficult to see that it does satisfy your differential equation, excellent.

So, I would advise you to do that diligently. Take a derivative here and verify that it satisfies the differential equation. One thing is obviously a plug in t0 instead of t here, the exponential term is in fact 0. Therefore, x of 0 is one and so x t0 is x t0. That is it is verifying that initial condition and after taking a derivative, I can also verify that it satisfies the differential equation.

Therefore, this is indeed the correct solution, the unique solution. This is important to us. This is a unique solution. So, what is the equilibrium in this case? What is the equilibrium? We did not discuss that. Let us see which way to get rid of this. So, what is the equilibrium, x sub e is 0, so 0 is the equilibrium because you can see the right hand side is 0, exactly at x equal to 0 for all time, so 0 is the equilibrium. So, I can succinctly write this quantity as this.

Now, what have I done here, what I have done here is I have denoted I have defined the quantity gamma as x of minus 6 sine t0 plus 6 t0, cosine t0 plus t0 squared, so notice, so this is what I have done, I have defined a new quantity gamma, which is just taking the initial time component of the exponential pulling it out. And notice that this is a constant whenever t0 is fixed, once I fixed the initial time, gamma is a constant. So, therefore, I have written it as such.

So, I have x t0 multiplying gamma, and now only a x exponential containing only a function of the time, so again, let us go back to our remember, we talked about this abuse of notation, you see that the solution does depend on initial time and also on t and also on initial state.

However, when we sort of write the left hand side, we just say that it is a function of time t, so this is an abuse of notation. But we have already understood what this is and therefore we are okay with it. But we do remember in our minds that the right hand side does depend on the initial time as well as the initial state.

So, once I have this sort of assignment, this definition, and I have this relatively succinct expression for x of t solution, I want to remind myself that there exists some finite time capital T beyond which this is negative, why? Why is this because t squared dominates 6t cosine t and 6 sine t. This is just a property of the functions involved.

If you look at this minus t squared, this is a quadratic term in t, so it is going to grow really fast whereas this one is a linear term in t with a cosine, which is of course going to fluctuate only between plus or minus 1.

So, it is just a linear increase in time and this is just a sinusoid term which is going to stay between plus minus 6 and this is of course going to stay between plus minus 6t. But this term is going to rapidly increase as t squared.

So, beyond a certain time, you can be sure that this function is going to become negative because this is a minus sign attached to the t square. So, therefore, there exists some capital T such that beyond that time, this quantity is negative, so this is a rather critical thing for us. So, what do we do?

We take this time, and we find the supremum of this function, what is it? So, we say that M equal to 0 and we denote M as this is actually a definition. We denote M as the supremum from initial time to cap T of this quantity 6 sine t minus 6t cosine t minus t squared. And why do I find the supremum only over this time?

Because I know that this quantity is possibly positive, only in this window. Only in this window, is this quantity possibly positive, beyond it, it is definitely negative.

So, if I try to plot this function, just to make things more clear, if I try to plot this function I can expect it to look like so, this is the time cap T is and this is this function right here on the Y axis. It should be evident that at t equal to 0 what happens? Let us see, in fact, let me actually be careful. I should not have done that.

So, let us say that this is t equal to t0, at t equal to t0 say it is some positive quantity. I do not know what it is. So, this is as you can understand this term is exactly going to be 6 sine t0 minus 6 t0, cosine t0, minus t0 squared. That is what is this guy. And then beyond that, say it is positive for some more time and then I have this peak. This peak is what is M, then drops.

And I know for sure that beyond this time cap T, it is definitely becoming negative. Therefore, I need to consider the supremum of this function only in this window. If I want to find the supremum maximum value of this function for all time, if I want to find the maximum value of this function for all time. I need only to look at this window t0 to capital T. And that is what I do. That is what I do. And look at only this window t0 to capital T.

So, I know that this function is definitely below M. Now what I claim is that if I choose an if I am given an epsilon positive, I claim that a valid delta is this guy. I claim this is a valid delta, why? Why is that? This is true, because if you assume that your initial condition is delta away from the origin, what does that mean? That means that my absolute value of x0 is less than delta, which is equal to epsilon over gamma e to the power M. Then, what can I say about the solutions?

Let us look at the solutions right here. I am going to write right here. So, absolute value of xt, I am taking the absolute values on both sides, and I am going to freely use the triangle inequality. So, this is going to be equal to gamma x t0 x of this whole thing, so, absolute value of this whole quantity, but I know for sure that this is less than or equal to absolute value of gamma, absolute value of x t0 times the absolute value of this whole guy by my usual triangle inequality.

Now I know for sure that this guy is upper bounded by M and because of what I just, how M is defined this guy is upper bounded exactly by M. So, what do I know? I know that this is less than equal to also notice that gamma is basically an exponential of some quantity. So, gamma is basically exponential of some quantity therefore, it is always positive so, I can get rid of this then I have absolute value of x t naught and this is capital M, this is capital M.

Now, once I have this you see, so, this I mean I have not been this is not very carefully done here this is actually this should be x t0 that is what we are using it is x t0, this is what we have been using.

So, once I reach this point that x of t is less than gamma M x t0 that is here, I can substitute for absolute value of x t0 which is chosen to be less than delta this guy.

So, once I substitute this here, what do I get here, the gamma eM cancels out and I get the xt is less than epsilon exactly as we required exactly as we require. So, we have been able to show that given an epsilon I can find a delta such that if my initial conditions start within the delta ball, my solution remains within the epsilon ball.

The important thing to remember is that the Delta and hence gamma depends on t0. So, in the definition of delta I have a gamma and if you notice the gamma was defined as such and this is dependent on t0 and therefore, delta is also dependent on t0.

So, what have we shown until now, we have shown that until this point if I see you be careful here until this point we have shown stability of 0 in the sense of Lyapunov. Now, what I want you to notice now, so we have already shown stability. So, the next question, of course, is it uniformly stable? Because those are the only two definitions we have done.

So, those are the only two things we can in fact test, so the next question is, is it uniformly stable? So, what I would like you to notice is that the only dependence of delta to t0 comes from gamma, that is the only way, so, we look at gamma itself. So, we know that gamma is this expression.

Now, what can I say about gamma as t zero goes to infinity as t0 becomes really large, you can see that gamma also goes to infinity as t0 goes to infinity gamma goes to infinity. This was sort of obvious and expected because the sine of t square is opposite in these two terms, which this is very standard when you do a definite integral. So, what happens is t0 squared again the quadratic term dominates every other term.

So, if I push my initial time larger and larger, then my t0 square dominates everything else in the exponential also goes to infinity, in fact very fast. So, if t0 goes to infinity, gamma t0 also goes to infinity. So, this is something you should remember.

Now, what we are claiming is that we cannot choose a uniform Delta independent of t0. And why we say that, we say that because as t0 becomes large gamma goes to infinity, so let us see.

Let us see why this is, let us try to understand just by some crude example, why this is? What am I saying? I am saying that the delta we chose was epsilon over gamma e to the power M.

Let us see, I am sorry, today, I think there was a slight error here. This is not gamma m. I am sorry, here it should not have been gamma M, but this should have been gamma e to the power M because the exponential was there. So, that is there was an error, that is okay. Now, the delta we chose was epsilon over gamma e to the power M.

Now, what are we saying that we cannot choose a uniform delta for all initial time. So, why are we saying this, we know one thing for sure that as beyond a certain time beyond say t0 greater than or equal to some cap T gamma of t0 increasing function gamma of t0 is an increasing function, this is not difficult to again claim, why, because, if you see, the t0 squared is dominating to beyond a certain time, this entire quantity, this entire gamma quantity, this entire quantity in the exponent of gamma is going to become positive and remain positive. And this will continue to increase.

So, it is going to be an increasing function. Another way of looking at it would be to sort of try to take a derivative of this exponent and confirm it, which is also not a difficult thing to do. So, if I take the derivative, also, I can confirm, so what is the derivative here very quickly. This will be of, of this exponent. So, I am trying to take a derivative of this guy, d dt is 6 cos t0 minus x cos t0 plus 6 cos t0 minus 6 t0 sine t0 plus twice t0. So, here, these two will cancel out. And I am left with just this much, twice t0 minus 6 t0 sine t0.

So, let us try to see what happens here. Is this going to be always increasing? Let me be careful here before I came this, this is going to be t0, sorry, t0. So, this is going to be t0 times 2 minus 6 sine t0. I see that this is not evidently increasing, is it? Let us see if I have miss something here, 6 cos t0 plus 6 cosine t0 minus 6 t0 sine t0, that seems about right, plus twice t0. And this is going to sort of be subtracted by this quantity.

No, this is not evidently increasing unless I am missing something. So, this is something I need to verify. So, in any case, we are sort of almost at the end of the time for this lecture. So, what we will do is we will sort of continue our discussion on whether this is uniformly stable or not in the next lecture so anyway what have we seen today.

So, what we have seen today is the definition of uniform stability which is slightly different from stability in the sense that it is uniform with respect to the initial time. The delta does not depend on initial time. And we started to look at an example a very nice example where we could actually solve the differential equation and verify stability at this point. We are now trying to check whether the system is uniformly stable or not. And this we will complete next time. So, that is it folks for this time. We will meet again soon. Thank you.