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Nonlinear Adaptive Control
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Week-2
Lecture 12
Notions of Stability-Part 2

Hello, welcome to yet another session of our NPTEL on Nonlinear Adaptive Control. I am Srikant Sukumar from systems and control IIT Bombay, we are again, in front of our motivational image, we are already well into sort of learning about tools and techniques that will help us analyze algorithms that drive autonomous systems such as this rover on Mars. So, let us see what it is that we spoke about last time.

So, until last time, we were sort of beginning to look at the notions of stability in the sense of Lyapunov, I hope I was able to impress upon you that this contribution by AM Lyapunov of Russian mathematician in 1900s sort of changed the world of nonlinear control. And honestly, there would not have been this sort of serious curriculum in nonlinear systems and control if it was not for the contributions of this gentle yes.

So, what did we do, we sort of began by looking at you know, what the system in consideration looks like. And we carefully stated that you know, this system will be always defined using a differential equation such as this along with an initial condition at a given initial time, alright? So, this was important to specify the system in completeness.

Now, we of course, also spoke about you know, the assumption of existence of unique solutions. So, we actually gave some examples, where we sort of saw that there is a possibility that solution may not exist for a given differential equation with initial condition beyond a certain point in time or it could also be that there is non-uniqueness in solution given an initial condition.

So, there could be multiple solution trajectories such as this picture here. For given differential equation and a particular initial condition, so, we want you to avoid these sort of pitfalls, at least in the discussion that we are having in this course. So, we make use of a rather nice assumption, right, which helps us to sort of evade these pitfalls, if you may, so, the assumption is that, this function f is Lipschitz continuous in the state x and it is continuous in time.

So, this is the assumption that we sort of ended up with we even saw that this is a sort of a sub linearity assumption. Yes, and, and, we saw that the cases that we considered, well, at least I hypothesized that the cases that we considered not fall under this Lipschitz assumption, and while I asked all of you to verify whether that is the case, and I hope, at least some of you put an effort into verifying whether you know, these functions are satisfying the Lipschitz condition or not all right, excellent. So, now that we have the system set up in place, right, that is this sort of differential equation with an initial condition right, once we have this setup in place, we are able to speak about what is an equilibrium.

So, equilibrium is any point that is any state x sub e such that f of t xe is equal to 0 is identically equal to 0 for all time greater than equal to t0. So, I will repeat the equilibrium is a state particular state xe such that f of t xe is identically equal to 0 for all time t greater than equal to t0. Now, why is this valuable or an important sort of point? It is because if you see if ft xe is equal to 0 what do I have?

I will have x dot equal to 0 for all t greater than equal to t0 correct. And if this happens, what does it mean? It means that my state never moves from xe. So, if I start in xe, if I start my initial condition at x sub e, then I remain at x sub e. So, in so this is to say that the point x sub e is in fact the solution of the system or trivial solution of the system if you may. And therefore, this is called the equilibrium. Yes, makes sense.

A lot of you might have actually seen the notion of equilibrium in physics. Yes, so you talk about the equilibrium of a pendulum, right? So, if you do this sort of settles here. Yes, it settles in the downward position and this is an equilibrium. Why? Because, if I start here, it never moves from here. This is what it means to be in an equilibrium.

So, we have spoken about you have definitely seen equilibrium in your during our high school physics classes. So, this is the notion of an equilibrium not very different from what you have seen in physics, yes. So, it is a state from which the system does not move ever unless there is a disturbance we do not assume any disturbance.

Therefore, the state never moves from the equilibrium, and that is what this condition is aimed at ensuring, excellent. Then there is the notion of an isolated equilibrium. And equilibrium is said to be isolated if no other equilibrium exists arbitrarily close to it. So, this is from definition 342 in the book Ioannou and Sun which is one of your references.

Now, I can of course make it more formal, we want to make it for formal. What would I say? I would say that there exists epsilon positive such that for all x belonging to b. Since we know the notion of a ball in a metric space, so this is the ball of radius epsilon around xe. So, ft x naught identical. But I will say if tx is not equal to 0 for all t greater than equal to t0, yes, what does it mean when I say something like this? It simply means that x but let me be more careful. I was See, I want to exclude Of course, yes, I of course want to exclude the point xe from this ball itself, because xe is of course, an equilibrium what am I saying?

I am seeing that any x in this small ball around xe is not an equilibrium, all right? This is what this condition ensures is not at equilibrium because it moves away from this. So, it is not equal to 0 for all greater than equal to t0, right. So, what does it mean it means that there is a ball there is a ball of radius you know, epsilon around an equilibrium and is within this ball every point other than the xe is not an equilibrium.

So, that is what it means to be an isolated equilibrium. So, it is very easy to construct examples of isolated equilibrium. So, I will actually give you one such example, of an isolated equilibrium, right? So, what is an example of an isolated equilibrium? Isolated equilibrium. And what is an example? Let us see, I can I can just construct something on the fly. So, this is x1 dot equal to x2 and x2 dot equals minus x1 right, if you do not like the linearity of this, I will just make it x1 squared. All right, no problem. And just make it x1 squared.

So, if you look at this system, what is the equilibrium? So, if I write it formally, what would be the equilibrium, the xe is equal to the set of all x1, x2 in r2 right, such that such that x2 comma minus x1 squared is equal to 00. So, there is no time involved, therefore, I am not saying for all time greater than equal to t0, because there is no time in the right-hand side.

No time appears here. So, whatever I do is always true for all time greater than equal to t0, time is not there. So, now if I equate these two separately to 00, this is equivalent if you may 2 x1 comma x2 equal to 00. So, these two are in fact equivalent. Therefore, the equilibrium xe is just the point 0 comma 0 in r2. So, this is an example of an isolated equilibrium, because there is no other equilibrium anywhere nearby.

If I want to make it slightly more complicated, because here, it looks like there is only one equilibrium, so why even talk about isolated and non-isolated? So, let us look at one more example. Yes, so let us see, it is x1 dot is say sine of x2 and x2 dot is well, let us see. I am not going to do that. I am going to just say x1 dot is x2 and x2 dot is sine of x1.

What are the equilibrium here? It is again the set of x1, x2 in r2 such that x2 comma sine x1 is 0 comma 0. So, what is this? Yes, this is not that obvious. So, this is actually sine x1 equal to 0 happens at all n pi. So, you actually have so, you should actually compute sine x1 is 0 for all n pi. So, this is actually of the form n pi comma 0 with n being an integer. For any integer n, sine n pi is 0. Therefore, all of these are allowed.

But you see these are still isolated. Why are these isolated should be obvious to you right? On the so if I draw this right, so on the y axis, this is 0, so all the equilibrium are on the x axis. So, one of the equilibrium is here. Second one is pi. And other one is minus pi, and so on. So, that does exist, you know, this ball that we want. This mythical ball of radius epsilon. and this ball of radius epsilon.

This ball of radius epsilon does exist, within which there is no other equilibrium, either none of these equilibrium except this guy. So, 00 is an isolated equilibrium, or minus pi 0 is an isolated equilibrium plus pi 0 is an isolated equilibrium. All of these are in fact isolated equilibria, and this is nice we like this property.

Let us look at another case of example of a non-isolated equilibria. And look at this example of a non-isolated equilibrium. So, what is it? It is a system which looks like x1 dot is x1, x2, and x2 dot is x2 square. So, if I write out carefully, xe is the set of x1 comma x2 in r2 such that x1 times x2 comma x2 squared is 00. Now, if you look at this carefully, this in order for this to be satisfied, I definitely need from here I definitely have x2 to be 0, but if x2 is 0 I also see that x1, x2 is 0 irrespective of what is x1 irrespective of what is x1. So, this is x1 is arbitrary.

Because if x2 is 0 because of the second guy here, but if x2 is 0 already then x1, x2 is always 0 respective of what is x1, so, x1 can be arbitrary, so, this is actually not correct, this is not correct. So, your xe will be of the form x1 or let me just call it some alpha comma 0. So, this is all alpha in real's are allowed. Now, what is the problem? Let us look at this equilibrium.

Let us look at the set xe, because x is not one point but it is a set alright. So, let us look at the set x what does it look like? So, I have the axis. So, this is the x axis and this is the y axis if you may order I mean in the notation of this paper so, the in the notation of this example is the x1 axis and this is the x2 axis and what do my equilibria look like, they have y equal to 0 in both cases okay and x can vary x can be anything.

So, my equilibria actually look like these I believe we actually look like this they span the entire x axis. And you can see this is not isolated. So, this is all of x axis and this is not isolated, why is this not isolated? Because, you see the equilibria really exist arbitrarily close to each other. There is no way I can separate them with a ball of epsilon, no way you cannot draw a ball of any size epsilon and miss other equilibria here because you will catch the other equilibria on the x axis no way you miss them. And this is a problem why do we care about isolated equilibria is because all of our Lyapunov stability definitions are for isolated equilibria only.

So, this is the critical point okay, this is the critical for isolated equilibria only. And why is that why is that see all the notions of stability are defined using norms and metric spaces. So, suppose I have an equilibria say denoted by xe and I take a comparison point x minus xe and take the norm because all the notions of stability in everything is defined using norms and this is why we introduced the notion of norms because so, that we can compare points and figure out how far they are from each other. So, that is why we needed the notion alright.

So, all the notions of stability based on norms. So, stability notions based on norms, and what does this mean? It means that whenever I evaluate x minus x sub e right, and the equilibria is not isolated okay. Equilibria is not isolated, then you can very well see that x minus x sub e plus epsilon yes, is almost equal to this. Yes, where epsilon is arbitrarily small. Yes, if epsilon is arbitrary small. If epsilon is arbitrarily small, then this norm and this norm are almost the same. Yes.

And so, what happens is that because my equilibria are really close to each other, in fact, there is no space between them. Yes, there is no way I can talk about stability of a particular equilibria because there is another equilibria really close to it and when I tried to compare between you know of compared x with some point here and some point here, it is almost the same. In fact, they will be the same if your epsilon is 0 or if epsilon is you know, you know 1e minus 10 1e minus 20.

So, you can see that in the absence of isolated equilibria it is difficult to even make sense of you know which equilibrium I am comparing with. And therefore, the Lyapunov stability notions do not work here. And therefore, we always assume that there exists an isolated equilibria for the system. So, I hope this makes sense to you yes, this is systems like these with equilibria like these are very difficult to define stability for okay and we will see immediately what these stability definitions look like and therefore, you will see that the norms appear and because the norms appear there is no choice but to assume some kind of isolated equilibrium.

So, let us see, let us see the first stability definition, this is called Lyapunov stability or stability in the sense of lyap, what does it require it is a test condition. It says that I give I get something and then I give something in return, it is sort of a test condition. So, what is this test condition for all epsilon positive, so, if I am given any epsilon, so, the user has to give me epsilon, remember, I cannot use choose epsilon is given to me by the user.

So, this is a common confusion a lot of students have whenever they try to prove the Lyapunov stability, epsilon is given by the user and hence arbitrary and this symbol therefore, the symbol for all epsilon positive, I must be able to find a delta which can depend on the initial time and this epsilon, which is also positive such that for all initial conditions with delta distance from the equilibrium the solutions remain at an epsilon distance from the equilibrium for all time.

So, let me sort of draw a picture to help us understand okay, so let us see. So, this is my axis, this is again a phase plane portrait. So, get used to this phase plane portrait thing. So, this is my axis. Now, let me draw two circles, so this is, I hope this looks like a circle I think so. So, the red one is the epsilon circle right, so, this is the epsilon circle and the blue one the blue one let me try to draw again, and the blue one is delta circle, no sorry. The blue one is the delta circle. So, what does this stability in the sense of Lyapunov say? What does it require?

It requires that if I am given the epsilon circle, so, they have epsilon circle is already given to me this is red ball is already given to me. So, this is the phase plane, so let me this is x1 this is x2 I am just showing, just for illustration we are showing it with, you know, in two dimensions because that is what we can illustrate very easily. In other words, it is difficult.

Yes, but otherwise this the same logic works for any dimension. So, what does it say? It says that, if you have a trajectory, which if you are given the epsilon ball that is this red's colored ball okay, then as a challenge answer, I have to be able to give you a delta ball. Now, this delta ball is has to be positive of course, so therefore, the actual ball metric ball right containing some points in the state space and it is allowed to depend on t0 and epsilon.

And what are we saying that any trajectory which starts in the delta ball yes, any trajectory which starts here stays within the epsilon ball for all time. Never escape the epsilon ball for all time. So, that is what does it say it says that if I start in the delta ball, I remain in the epsilon. So, that is the whole idea. So, this magenta thing is the trajectory of the potential trajectory of the system. So, any directory which starts inside the delta ball must remain inside the epsilon ball. And this is what it means for the system to be stable in the sense of Lyapunov.

Now, again, let us go back to this issue of non-isolated equilibria. And because we said that we cannot deal with them, and let us try to see why, now, we are talking about this xe. So, I mean this of course, it should be obvious to you that the point in the center here, what is the origin in this case is xe is the equilibrium.

Now, in the case of non-isolated equilibria, you can see that very close to here, I can have another you know, I can have another equilibrium, or another equilibrium very close. Now, if it is this close, what do you expect will happen, you can see that I can replace xe by any of these points in pink. I can replace xe by any of these points, these things on the sides. Yes, and nothing much will change.

So, the problem becomes that, but the thing is, this circle, of course, has to be centered on the other one, but because it is a norm, these values are not going to change significantly, especially as this pink thing comes closer and closer to the blue point and that is a problem, because I do not even know which equilibrium, I am talking about. I may be talking about xe, or I may be talking about these pink things. I do not even know which equilibrium I am trying to test the stability of, and this is why dealing with non-isolated equilibria is a rather difficult challenge. So, we of course, deal with it in different ways. So, we will talk about it later as and when we need it. Yes.

Now, just remember that we require the equilibria to be isolated. Now, one of the things that should sort of be evident to you is that delta, delta has always has to be less than or equal to epsilon. The way this definition is made, you require the delta has to be less than equal to epsilon. What happens? Let us try to you know, do a thought experiment, if delta greater than epsilon, what could happen? If delta greater than epsilon what would happen?

So, what I am saying is that x0 minus xe is less than delta. But this is greater than epsilon. So, when I say that, I require xt minus xe to be less than epsilon for all t greater than equal to t0. This has to of course be true at t0 also. This has to be true also at t0 because notice, I said it has to be true for all t greater than or equal to t0.

So, if I plug in t equal to t0, this still has to hold. But now notice, I said that this is less than delta which is larger than epsilon. So, there is no guarantee that at initial time this is less than epsilon at all. It may be but it may not be either, therefore, the challenge fails at the initial time itself if delta is greater than epsilon. So, by mistake if you get an answer which says that delta is greater than epsilon, then you can safely assume that that was the wrong answer. Great. So, excellent.

So, what did we talk about today, we sort of ventured forward to look at what is equilibrium, what is an isolated equilibrium and what is a non-isolated equilibrium and why an isolated equilibrium is rather critical for talking about stability in the sense of Lyapunov. And then finally, we saw the first stability definition that is, in fact stability in the sense of Lyapunov. And we saw the first epsilon delta definition in this class. So, this is where we will stop, but of course, in the future also, we will see further stability definitions and more epsilon delta definitions. Thank you, folks.