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**Nonlinear Adaptive Control
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Week-2
Lecture 11
Notions of Stability-Part 1**

Hello folks, welcome to yet another session of our NPTEL on Nonlinear and Adaptive Control, this is Srikant Sukumar from systems and control IIT Bombay. So, we are again back with our motivational image of this rover on Mars and be one to use this sort of background image to help us remember that the algorithms that we are trying to develop and analyze are eventually to support systems such as these to be driven autonomously. So, without any further delay, let me sort of give you a little bit of recap on what we were doing until last time and if you plan to do now.

So, before today, we were talking about this notion of Barbalat's Lemma, we give you know, sort of two different versions of this lemma, right? So, we sort of spoke about this integrability condition. And we also spoke about a corollary which contains which is in terms of this L infinity, and L_p functions and so on and so forth. Alright, and then in order to illustrate how useful this tool is, we started talking about the spring mass damper example.

The important thing to note, in this example, and the way we did this analysis was the sort of standardization of the steps. So, the steps starting from this, you know, step number 1, right here were essentially very, very standard steps in our analysis, and what I sort of reiterated again and again, is that these steps are going to remain identical every single time we try to use the Barbalat's Lemma to do the signal changing analysis.

So, this is what I was sort of hoping that all of you would remember for sure. So, so, these are very, very standard steps. And then you know, we also you know, saw how you move from proving the terms in the \dot{v} go to 0 to proving that the other terms that is the terms that are not in \dot{v} also go to 0 using the original Barbalat's Lemma.

So, so, we sort of claim that this is a very very generic or more general-purpose theorem, which can be used to prove convergence. So, this was another thing we stated very carefully, at the end of this analysis, we can prove that our signals converge to 0 and further that our signals remain bounded. So, these are the two things we prove, we have not yet spoken about stability, but we will do so, in the subsequent lectures.

Yes, and we also said that this is more general purpose and application than something like the LaSalle invariance principle in its you know, classical form. Yes, and in order to illustrate this is why I have given this exercise number 4, where you have these time varying gains in the system now, and the Barbalat's Lemma is still applicable to these systems in order to prove convergence of signals boundedness of signals.

So, systems such as 3.25 are not analyzable using the Lasalle invariance principle. So, now that we have seen the Barbalat's Lemma, and of course, the other associated very very critical lemmas we should keep in mind that Barbalat's Lemma was not the only result that we saw.

We also saw these other lemmas on you know the existence of bound of functions and the lemma on uniform continuity of functions. So, we had these two other rather critical lemmas also appearing Alright. So, great, great.

So, now that we have seen the Barbalat's Lemma, what we want to do is we want to start looking at the notions of stability. So, now please do not worry this says week 3 lectures, but we are still only in week 2. So, it does not matter, we are advancing a little bit into the week 3 lectures also right now, because in the future, we anticipate more the material to get more involved and therefore, if we advanced further into the later week lectures, there is no issue. So, what we want to talk about now is the notion of stability in the sense of Lyapunov.

Now, before we actually delve into stability, we sort of want to give a setup of what kinds of systems we are talking about. So, of course, we say stability in the sense of Lyapunov and this is due to this is due to AM Lyapunov. And of course, somewhere 1800s and mid 1800s is when he came up with these sort of stability notions and also gave useful tests for stability.

So, rather critical, I would say rather critical without the contribution of Lyapunov of our entire field of nonlinear system will be in a very, very primitive state, without these results, we will not have been able to analyze the stability of nonlinear systems, we will not have been able to talk about tracking. So, therefore, there will be no question of you know, I mean trying to understand how systems perform under feedback. So, this this is a seminal tool in nonlinear systems, any nonlinear system.

So, any nonlinear systems course or the adaptive systems course is incomplete without the study of the Lyapunov stability motions and the theorems. And so, that is of course, what we intend to look at. So, the typical system that is in concentration is a standard state space system, it is $\dot{x} = f(x, t)$. So, f is the you know, vector field which is determining how the system evolves over time. So, it is and you expect this to be a function of the state as well as the time so, as all of you know. I assume x are the states and t is the time and the system is not specified unless and until I also give an initial condition, what is the meaning of an initial condition it means that I specify at an initial time t_0 an initial state x_0 okay.

So, without this without you know this initial condition the system is not considered to be fully specified. The other thing to remember is that we always assume that this system that is this differential equation with these initial conditions has a unique solution. And we have not discussed this here at all, but what does it mean, for a system to have a unique solution, it means that given this initial condition there can be only one solution possible that exists.

So, first of all, that there exists a solution okay. So, the first thing is that there has to be a solution it should not be that I take an initial condition there is no solution beyond a certain time that would be a problem where there exists a solution and secondly, this solution is unique. Okay. Secondly, this solution is unique. So, this is rather rather important.

So, one of the things to sort of, you know, I mean remember is what is what is the condition for existence of a unique solution. So, this means that $f(x, t)$ is Lipschitz in x . We need $f(x, t)$ to be Lipschitz in the states, this is the requirement for solutions to exist and be unique. So, now what is this Lipschitz Condition? Lipschitz in x and continuous in t , we will actually Lipschitz in x and piecewise continuous in t is fine, but I am going to say continuous in time. I am going to make it a little bit more, you know, relaxed.

So, so what is, you know? So, of course, we are saying so many things, so we want to understand where sorts of things can go wrong. So, let us see, let us consider an example. Yes, say this system of the form $\dot{x} = x^2$. And let us try to integrate and see what happens. So, so you are of course given the initial condition of the form that is this. So, if I try to solve the system, what do I get?

So, I get $\frac{dx}{x^2} = dt$ and of course, I integrate from t_0 to t on both sides are actually on the left-hand side this will be $\frac{1}{x} - \frac{1}{x_0}$, so, what do I get here? I will get something like $-\frac{1}{x} + \frac{1}{x_0}$ yes, so, this is yes, this is just using the standard integration formula for $\frac{1}{x^2}$. Yes, and this will be $t - t_0$. So, if I actually solve for this what will I get for $x(t)$? This will be let us see, let us look at this carefully this will be $\frac{1}{1 - x_0(t - t_0)}$. I believe this will be the solution. So, yes.

So, this will be the integration of the left-hand side and this is the integration of the right-hand side then I take this guy to the right and this to the left. So, I get and then I flip this yes, then I do this take the inverse. So, I get something like this., absolutely. So, look at this what happens to this system, what happens to this system? Let us try to understand. If you see, so, if $t - t_0$ is equal to $\frac{1}{x_0}$, $x(t)$ blows up, $x(t)$ blows up. So, basically bad things start to happen, so black bad things start to happen at $t - t_0 = \frac{1}{x_0}$. And this is called finite time, escape.

So, the solution seems to exist until a certain time. It is until the time is right less than under the time elapsed that is $t - t_0$ is less than $\frac{1}{x_0}$. But at $\frac{1}{x_0}$ the solution blows up. So, solution does not exist anymore. So, if I sort of changed this a little bit I mean if I if I instead of doing this if I even change this a slight bit that is I made this say $\dot{x} = x^3$ instead of $\dot{x} = x^2$ with some initial condition as before and I follow the entire set of sequence of steps as before then what I will get, is my solution will be $x(t) = \frac{1}{\sqrt{1 - 2x_0^2(t - t_0)}}$, let us see minus.

So, this will be square root of 1 over, let us see I want to be careful here. So, let me actually write it down so, that I do not make a mistake if I make this then I will integrate So, I will get from here $-\frac{1}{2x^2} + \frac{1}{2x_0^2} = t - t_0$ and from here I will get $x(t) = \frac{1}{\sqrt{1 - 2x_0^2(t - t_0)}}$.

So, this is what you will have. So, why did I make this slightly different today I want to look at the slightly different case because, if you look at this case if you look at this slightly different case, what happens is that here in this particular case, yes, if I cross $t - t_0 = \frac{1}{x_0}$ if $t - t_0$ is larger than $\frac{1}{x_0}$, solution still again start to exist. So,

solution only does not exist at that one point that is $t - t_0$ equal to 1 by x_0 . But as soon as I change the power to x cubed for example, then what happens is I get a square root here.

So, if I look at what happens in this case is you there. So, in this case, no solution beyond $t - t_0$ equal to 1 over twice x_0 squared. There is no solution beyond this because after $t - t_0$ becomes larger than 1 over $2x_0$ squared this entire thing becomes a negative quantity and so you start getting an imaginary number okay. So, solution exists only until this point and never nothing beyond.

So, this is a problem system we do not want to consider systems of this kind. So, this is a case where solutions do not exist or they have what is called a finite escape time. So, we do not like these kinds of systems.

Another example of systems that we do not like is the ones that have non-unique solutions, what are such systems? So that is, so this is this was non-existence, they now I have non-uniqueness. Okay, so what are such systems? So, let me look at one example only, suppose I take \dot{x} equal to square root of x , there I had the square here I have the square root, so let us look at what happens. And here I get x at 0 equal to 0 , I give initial time as exactly 0 and initial state also as 0 .

So, let us see what happens. So, again I do integration as always. Yes, so this is now of course, 0 to t and this is 0 to x . So, what do I get? I get yes, what do I get? So, the here I get So, this is x to the power minus half. So, when I integrate I get $2\sqrt{x}$ equals t . So, which implies x equal to t by 2 whole squared. So, this was x to the power minus half so, when integrate I get x to the power plus half divided by half so, that is $2\sqrt{x}$ and this is equal to t on the right hand side. So, initial condition is initial time is 0 and the initial state is 0 , so, I get this.

So, using this initial state 0 I have 0 here initial times 0 , I have 0 here. So, so, now notice that this is definitely one solution that I found out of you know, legitimately integrating this thing, so, I found it legitimately integrating this quantity. Now, however notice that x equal to 0 is also a solution yes, that is x staying x exactly at 0 is also a solution. So, if I make a picture for example, as to illustrate what will it be, so, if I look at the t and have a look at x of t the amazing thing is that there are two solutions. So, x equal to 0 is a solution yes, fine.

Let me not do this, let me do it in a better way. So, that we can actually see things yes, on the x axis this time and on the y axis I have the x of t then there are two possible solutions right. So, one is this this you know x equal to 0 solution and the other is t squared by 4 which is some kind of hyper like a parabola, the other solution is again parabola.

So, it will be something like this. So, you have two solutions x equal to 0 and x equal to t squared by 4 . So, this is an example of a non-uniqueness of a solution. So, both these cases are rather bad. Because in in that earlier case, there was no existence of solution at all. So, that is, of course, very bad.

In this case, what is happening is that, at the starting at the same initial condition, I can possibly get two paths in which the system can move. Now, you can imagine analyzing stability for such

systems is really complicated, really difficult in the first case, because I do not even know if solutions exist beyond a certain time. So, I cannot guarantee anything about System Properties beyond a certain time. And in this case, you know, I cannot even talk about stability because, you know, the at a particular initial condition paths seem to diverge. So, I do not know how many paths there are and what happens on those fronts. Yes, there are modern notions of stability, which do deal with systems like this, but in our case, we do not consider such cases.

So, what do we need? We need the like I said, we need the Lipschitz condition. So, what is the Lipschitz condition, Lipschitz. Condition is sort of a sub linearity condition, I am going to state only the global version, local version, you can look up there exists a positive l such that f of x minus f of y in the norm is less than equal to l times norm of x minus y . So, this is what it means for a function to be Lipschitz in the x state argument.

This is what this is called the Lipschitz continuous sort of a sub linearity condition. So, it is not very difficult to see that in both these cases. This kind of a sub linearity condition will be violated; I would strongly encourage you to check that this sort of a Lipschitz condition is going to be violated yes, for the examples that I have actually spoken about. Excellent.

So, what is the setup, we have a system a nonlinear system, we have an initial condition and we usually denote our solutions by this. So, this is sort of a I would say, abuse of notation, strong abuse of notation, if I may, the solutions in general are always denoted by ϕ of t, t_0, x_0 . Why? Why do we do this is because the solution is depends is depending on time, that is obvious, but it also depends on the initial time and the initial state, which is pre specified here. If I change the initial time and the initial state, this definitely changes. However, for the purposes of you know, keeping our notation under control we, what do we do, we actually just call it x of t , just like we did here in all our examples, we called a solution as x of t .

In fact, if you noticed, in all these cases, the solution was a function of t initial time and initial state. So, it was actually a function of all these three quantities, but we sort of suppress that in our expression just to keep things succinct, however, please do remember, please do remember that it is very, very critical to remember in fact, that we are always talking about a solution with a pre-specified initial time and an initial state. So, the solution definitely depends on the initial time and the initial state, there is no two ways about it, all right? Great, great, great.

So, I just want to summarize what we spoke about today. So, what we discussed today was really just the setup, for discussing nonlinear stability. So, stability of systems in the sense of Lyapunov is a rather critical notion and we have simply just started discussing the setup of this. So, what is this setup, it took us some time to talk about it, because it is rather critical, what is the setup, it is the idea that when you specify a system, first of all you need to provide certain initial time and initial states. So, what is called initial conditions?

The second rather critical thing is that, we need to sort of presuppose or presume that there exists unique solutions to this system, given this initial state and time. Now, in order for that to happen, we saw some counter examples of systems which either do not have a solution beyond a finite time or they do not have unique solutions, given a certain initial condition, we want to

avoid both these sorts of systems and therefore, we assume something called Lipschitz continuity of the vector fields that is the $f(x, t)$ on the state x .

And we assume continuity in time. And both of these conditions is what helps us to have existence of unique solutions. Of course, we have not proven why this is the case that is slightly outside the purview of what we want to do. You can always look at the proof in standard references like nonlinear systems by Khalil, in order to find proof of the fact that if you have Lipschitz continuity then you will get unique. So, you get existence of unique solutions. But we simply go ahead and assume it and this will help us to talk about equilibrium and stability in the sense of Lyapunov, in the upcoming session. So, this is where we will stop. Thank you all for attending and let us meet again soon.