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Nonlinear Adaptive Control

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Week-2
Lecture 10

Convergence of Signals using Barbalat's Lemma – Part 2

Hello everybody, welcome to yet another session of NPTEL on Nonlinear Adaptive Control. This is Srikant Sukumar from systems and control IIT Bombay. As always, we are in front of this very nice background, this Mars Rover. And what we learned in this course, are the sorts of algorithms that drive systems like this autonomously without communication to any earth station on Mars and the Moon.

So, let us go forward with what we have been doing in this course until now. And so, we had started off with the talking about the Barbalat's lemma and its corollary last time, right? So, we had sort of discussed the corollary to the Barbalat's lemma and in order to motivate how to use this Barbalat's lemma we had started by discussing standard spring mass damper system, we of course, you know, wrote the dynamics of the system created error states corresponding to reference trajectories, wrote everything in wrote the error dynamics in state space form.

So, we wrote the error dynamics in the state space form, then, we created this target system that we wanted to achieve. And we did that by choosing a very specific feedback controller. It was, of course, evident to us by standard, you know, linear systems methods that this is a exponentially stable system. However, we want to prove the same using energy functionals like these, because this is what will eventually help us when this target systems turn out to be nonlinear, and of course, also in adaptive control cases.

So, what we did with Barbalat's lemma was only to prove asymptotic convergence. So, we cannot say anything about transient performance, bad things could happen in the transients, but we are only using the Barbalat's lemma to claim that, for large time, the errors converge to 0. So, we did a few steps. So, we took the derivative of the candidate potential function, and we found that the derivative was less than equal to 0 while the potential function was greater than equal to 0. And using this we started our signal chasing analysis, and we did a few steps.

The first step was to show that V infinity that is the limit as t goes to infinity, vt exists and is finite, then the second step was to show that all the closed loop signals, that is e1 and e2 in this case we are bounded. And the third in a rather critical step was to show that the signal e2 is in fact an l2 signal.

Yes, very quickly recapping this last point, because this is a very important point. How we did this was we integrated both sides of this v dot and it was possible to integrate the left-hand side only because we know that limit as t goes to infinity vt is finite that is v infinity is a finite quantity. And since we can do that, we have this right-hand side, which is essentially the integral of e2 squared.

And we also want to remind ourselves that the two norm of e 2 looks something like an equation 313, and we see that this is very similar to what we have here. And therefore, what we have is v infinity minus v 0 is just the square of the 2 norm of v2 multiplied by again k2 and so, from this we can compute what is the two norm exactly, and this turns out to be finite and real, because of the fact that v infinity is finite and v use non-increasing.

And this is essentially what is the definition of a signal to be an l2 that it is l2 norm v finite, okay. So, excellent. So, this is where we ran until last time. And so today, we start off with our lecture number 4, so we are on lecture 4 of the week 2. We are on lecture 4 of the week 2. So, what do we have here?

So, the next step is to prove that the derivative of e2 is l infinity. That is it is bounded? Yes. So, you want to prove that the derivative of e2 is bounded. Why do we want to do that? Well, I mean, if you all remember the statement of the Barbalat's Lemma and the corollary, you would remember that if a derivative of the function is bounded or l infinity, then we can claim that the function itself is uniformly continuous. And we know that uniform continuity plays a critical role in the Barbalat's Lemma. So that is essentially why we care about the derivative of the signal beam, l infinity, all right? Excellent.

So, how do we claim that we look at the derivative itself, so the derivative is rather straightforward in this case, it is given by this expression e2 dot is minus k1, e1 and minus k2, e2, you should remember that k1 and k2 are just some positive constants.

And we have already proven in step number 2 that e1 and e2 are bounded. And because e1 and e2 are bounded, we have that k1, e1 and k2, e2 are also bounded. And therefore, this quantity is also bounded so, excellent. So, now, if we use the corollary to the Barbalat's lemma, not the Barbalat's Lemma itself but the corollary. Yes, this is corollary 2.2. What did it say? It said that, if a signal is l infinity and sum lp and further its derivative is l infinity, then the signal itself goes to 0 as t goes to infinity. So, and that is exactly what we have here. That is exactly what we have here, we have that e2 is l infinity by the fact that e2 is bounded in step 2.

So, this I will write it, for this particular case. So, this is step 2, e2 is l2 was in fact, step 3, right? We prove e2 is l2 in step 3, and the fact that the derivative is l infinity was proven in step 4 okay. And with all these ingredients, we can invoke this corollary 2.2 here.

Let us go back again and look at the corollary just to remind ourselves if a function is in any l infinity and any lp and further if the derivative is in l infinity, then the function itself goes to 0 as t goes to infinity.

So, therefore, we have proved that limit as t goes to infinity, e2 of t is exactly 0. So, we have actually completed the proof of one part, or we have completed one part of the proof, I am sorry, yes, because we started by trying to prove that both signals e1 and e2 are going to go to 0 as t goes to infinity. And so, we have been successfully successful improving one part of it, okay. What is this part say in the context of the spring mass damper?

It says that your velocities Yeah converge to the desired velocity trajectories. We have already proved in 318 that the velocities converge to the desired velocity trajectories. So, moving on, so this is only one part of the proof. So, one half of the proof. Therefore, we would like to move on and try to complete the proof. And let us see how it looks.

So, in the next step, we want to prove that the signal e2 dot is, an integrable signal. Notice where we started. Notice where we started it, this is very, very critical again, because these steps are rather standard. So, you need to understand these steps very well. So, notice where we started in step 6. Until step 5, we have been able to prove that e2 goes to 0 and what exactly is e2.

e2 is the only quantity that shows up in v dot. So, the first half of the proof, essentially, lets you conclude that terms appearing in v dot converge to 0. See, this is what we will be able to prove in the first half of the proof of first half of the sequence of steps. Yes, that all the terms that appear in v dot not all the terms of course, I mean, we are doing an inequality, that so we essentially have all quadratic terms. Well, I do not want to say quadratic already, I am sorry; I do not want to say quadratic per se. But all the terms that, you know, you have sort of in the end in v dot, and you will see more examples of this, it will become clear, this is merely a guideline, it is not a law or a theorem.

So, if you see v dot in this case, it contained only e2. And so, in the first half of the proof, we are able to show that e2 goes to 0, then where do we start in the second half of the proof? Now, this is a important question. So, in step 6, we start exactly at the derivative e2. Until now, we have proven that e2 goes to 0. And now let us, let us all recall all these myths that we spoke about. Yes, we know that e2 converging to 0, does not necessarily mean that the derivative of e2 is going to go to a constant.

It may not even have a limit. We looked at several examples. I mean, really, you know? You know, it is rather easy to construct such examples. So maybe it is something like for example, sine t squared. Let us see, divided by t, this quantity has a limit as t goes to infinity. And so this is this was the example. But if I take the derivative, if I take the derivative minus sin t square over t square plus 2 cosine t squared has no limit as t goes to infinity. So, this is this was sort of yes, this funny looking box is sort of this you know, what we proved.

So, sin so just the fact that a signal goes to 0 does not mean that its derivative is going to converge. And so, this is exactly what we start to do in this next step. Yes, we start looking at the derivative of the signals we proved are going to 0, in this first half, we prove that all the terms appearing in v dot goes to 0. And now we start with the derivative of those signals. So, we prove that e2 dot e2 goes to 0. So, then we start with e2 dot in the next stage and this is important to remember.

So, I am going to write this start with derivative of converging terms from above. We start here with all the converging terms from above. So, the derivative of all the converging terms from above. So, in this case, this is only e2 dot. So, what do we show we show that e2 dot is integrable? Why? Just integrate it? How do we compute this simply by integrating it, if I take this integral right here, you see that, that infinity, so this is again, same idea as before, so you have.

This is d e2 by dt and dt, so this is just integral of d e2, which is essentially e2 at infinity minus e2 at 0. And again, why was this possible? This is possible because e2 infinity is nothing but 0. e2 infinity is the limit of e2 as t goes to infinity and that is 0 by the previous step. Otherwise, this would not have been possible if we had not proven that the e2 had a limit as t goes to infinity, then this step would have been impossible, okay. Excellent. Excellent.

So, I hope you understand that quite well. So then, because we have this step, this gives me that this integral of e2 dot dt as to 0 to infinity is just minus e2 at 0. And this is a finite integral because your error started at a finite value. I mean, it would not make any sense otherwise. So, this is a finite integral. And therefore, this essentially helps us conclude that e2 dot is, an integrable signal okay, e2 dot is an integrable signal. Again, you should remind yourself of the Barbalat's Lemma and its corollary because now, we are moving towards the original Barbalat's Lemma statement. Yeah, where we need integrability and uniform continuity, excellent.

So, what do we do for uniform continuity? Because we already have integrability. So, what do we do for uniform continuity, evaluate the derivative, as simple as that. So, we now claim that e2 double dot is an infinity, because, this will imply that e2 dot is uniformly continuous? Yes, because the derivative of e2 dot is bounded, therefore, the signal itself has to be uniformly continuous by the lemma we have seen before, excellent.

So, what do I do? How do I compute e2 double dot just take the E2 dot and you know take derivatives again. So, just take e2 dot here and I take the derivative again on both sides. So, this needs to be minus k1, e1 dot minus k2, e2 dot here and then I substitute for e1 dot and e 2dot using the dynamics again.

So, this is minus k,1 e2 and minus k2 times minus k1, e1 minus k2 e2, okay. And you will notice that all these terms again k1and k2 are constant so, they do not play any role in signals becoming unbounded no question of that happening. But what remains are just states here e1 and e2 and you have already proven in step 2 in fact long long time ago Step Two that e1, e2 are bounded. Therefore, e2 double dot also has to be bounded right, e1, e2 is bounded. So, the right-hand side has just bounded quantities. So, the left-hand side is obviously bounded. Yes, sum of bounded quantities or difference of bounded quantities is necessarily bounded, alright. So, we have essentially now shown that e2 double dot is l infinity, and this immediately tells me that e2 dot is uniformly continuous, excellent.

Now, let us go back and look at the statement of the original Barbalat's Lemma. What does it say? It says that if a signal is integrable and it is uniformly continuous, then the signal is going to 0 as t goes to infinity.

So, let us look at what we have. We have just shown that e2 dot is integrable and because e2 double dot is l infinity e2 dot is uniformly continuous. So, by the original Barbalat's Lemma statement, I can immediately conclude that e2 dot in fact goes to 0, all right? So, I hope this sinks in? Well.

We started after the end of the previous half of steps with the derivative of the terms that we had proven to be going to 0, which is only e2 in this case, in a more general setup, this could be multiple signals. Yes, e2, e3, e4, and so on and so forth. Yes, and in the next stage, we start with the derivative of all of those. So now, e3 dot e4 dot e2 dot e3 dot e4 dot e5 dot so on and so forth. And we prove there integrability. We know, why do we start with this step? Because we know this is immediately possible. Yes, because I prove that I have proven that all of these signals go to 0 as t goes to infinity.

Therefore, if I integrate it, it is simply going to depend on the endpoints. And this endpoint at these endpoints, that is an infinity and 0 signal is finite. And so we are done. And this is true for all signals. Yes, even if you have e2, e3, e4, e5, e100, does not matter; it will come out to be like this. As long as you have proven the these signals converges to 0. Excellent.

And in the next stage, we prove that e2 dot is uniformly continuous by proving by taking its derivative, and showing that it is bounded is also very easy. I can take many derivatives further. This will always be bounded. Yes, because you will always get terms in e1 and e2, which are already known to be bounded okay, excellent.

So, with this with the Barbalat's Lemma, we have proven that the derivative of e2 also goes to 0, right? So, so we have actually taken a nice formal procedure to prove that after we have proven that e2 goes to 0, we have made a nice, nice formal process to prove that e2 dot also goes to 0, we did not just conclude it, you know, just like that, all right? Great.

So, once I have e2 dot goes to 0, the rest of the proof is not difficult at all. Why is that? I immediately claim that e1 also goes to 0. So, look at the expression for e2 dot, it is minus k1, e1 minus k2, e2 and here I know, that the limit as t goes to infinity e2 dot is 0, and I know that limit as t goes to infinity e2 is 0. So, what do I do?

I can simply take limit I apologize; I simply take the limit as t goes to infinity of both sides, and they have to match because I have an equivalence here. Yes, if I say a is equal to b, a function a of t is equal to function b of t, then the limit as t goes to infinity cannot be different. Therefore, if I take limits on both sides, what happens I know that this guys is going to 0, I know that this guy is going to 0. So, the only way that the left-hand side can go to 0 is if this quantity is also going to 0 and k1 is just a positive constant.

So, this is possible only if e1 itself also goes to 0 as t goes to infinity. So, this is very, very important. So, what have we shown? We have shown with a bunch of 9 steps to be precise, yes, looks like a lot of steps. But when your problem gets more complicated and you cannot resort to linear analysis, or you have an adaptive system, which inherently makes your closed loop system nonlinear, these 9 steps will look like a blessing to you.

Yes, as of now, it looks like oh my god, I could just have computed the you know, Eigen values and be done in a moment. But we took 9 steps, but like I said, when problems become nonlinear and more complex, these 9 steps will be super easy, simple, significantly simple, or, in fact, just possible, okay, that the step would not even be possible. Excellent.

So, we have as we planned, we have been able to show that e1 and e2 both go to 0 as t goes to infinity. Of course, we could have used this Lyapunov analysis with LaSalle invariance; we have not done this still. Yes, in this particular case, we could also have used the Lyapunov analysis and Lasalle invariance. However, Barbalat's Lemma can be used in a larger scheme of things.

So, it is well known that this this will work only for time invariant systems okay, we are only for time invariant systems. Of course, there are modern versions, which were for some get some sort of time varying behavior, but they are not all encompassing. So, in general LaSalle invariance principle, which again, we are yet to cover works only for time invariant systems. So, if I have something like this, where my gains instead of being constant gains, their time varying gains, again, something that is not uncommon in adaptive control, then only Barbalat's Lemma can be applied, these cannot be applied anymore.

Here only Barbalat's Lemma can then be applied to prove convergence, okay. So, of course, this is an exercise, you are to prove convergence of e1 and e2 to 0 using the similar steps, any additional assumptions you are free to make. And you are but you have to state all these assumptions and to state all these.

Now, I want to remind you that we only proved convergence. So, we only been we only completed the proof of convergence of these signals to origin. So, we are saying something about the steady state behavior only, we are not saying anything about the transient behavior. So, remember, this convergence only has been proven stability, or the fact that the trajectories remain bounded.

See, one thing should be obvious to you already is that by step 2. Here let us see, by step 2 here, I mean, it is not like we have got nothing beyond convergence, we did get something more than convergence. We are also saying that e1 and e2 remain bounded, so all the closed loop signals, So, notice that our trajectories we chose were themselves bounded with bounded derivatives. So, what does it mean it means that if e1 and e2 are bounded, then x1 and x2 are also bounded. So, we do have convergence from this, but we also prove that all our closed loop signals remain bounded.

So, this sort of thing that you know, I gave this funny sort of example, this cannot happen, this cannot happen. Yes, this cannot go to infinity and come back, it can become very large and come back, but everything is still bounded.

So, all the closed loop trajectories are going to be bounded. And on top of it, use proven convergence. Okay, so that is sort of the good thing. So remember, that is what we have. So, we have completed the proof of how to use the Barbalat's Lemma to conclude asymptotic convergence of signals, stability is a separate question, which we will address slightly later.

So, what did we do today, we continued our proof of convergence of signals using the Barbalat's Lemma. And we had finished, you know, proving that a signal was l2, and then we proved that whatever terms appearing that appear in v dot, they converge to 0. That is the first thing we prove.

Then we start off our next set of steps with the derivative of those signals that we proved already go to 0. And then from that, we prove that these derivatives of those signals are integrable by virtue of their convergence, and then that the derivatives are bounded. And from this we can prove that the derivatives of signals that we proved were going to 0 themselves also go to 0.

So, if e2 goes to 0, we prove that e2 dot also goes to 0. And using the dynamics that is how the equations of e 2 dot look, the show, that, e1 itself also goes to 0, of course, later on we will also see where these things fail. Yes, and we get into detect ability obstacles in adaptive control. So, we will discuss that soon. So, this is where we conclude today. Thank you for your attention.