Advanced Strength of Materials

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Lecture – 8

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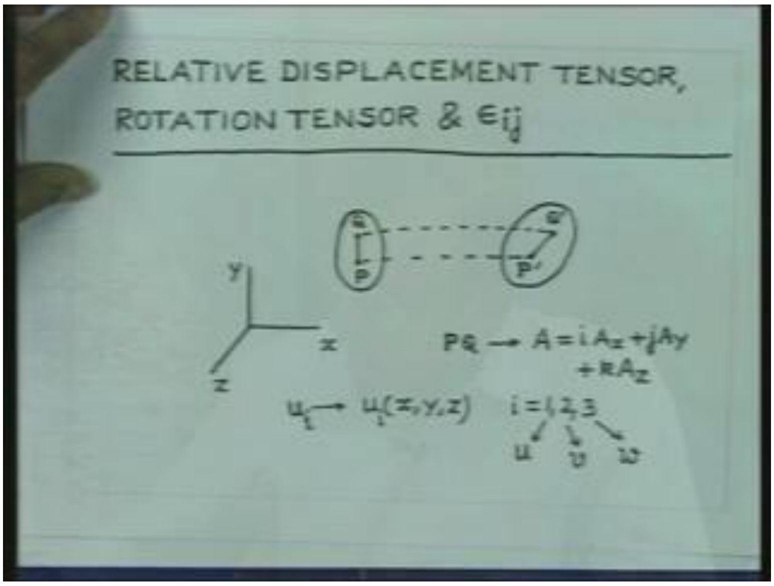

We would like to consider in this lecture, relative displacement tensor, rotation tensor and strain tensor. We have already seen earlier, how the strain tensor is defined, we would like to look into little more into the definition. And so that this strain tensor is related to relative displacement tensor and rotation tensor, when you consider any component loaded, it will undergo displacements. And those displacements can include the rigid body displacements, it can also include the deformation. So, let us try to look into specifically into the case, we have consider three dimensions.

And our axis are oriented as usual consider a body, which is loaded I am not showing the load here, that this is the body which is subjected to loading. And under the action of loading it can move it can rotate. So therefore, it can have deformation also it can, it is moving in space and it has both rotation and deformation. So, if you confine or attention onto two points, although we have showing it to be this two points to be widely apart.

But in fact, are interest to consider this two points, which are closed by... So, these are the two points. Now, are it moves in the position that we have shown here, it might take up orientation, which is given by P dash and Q dash. So, the point has move within space like this. Let us consider that the distance between P Q can be indicated as a vector quantity.

And let us say, that this is the vector A and this vector has components A x in the x direction A y in the y direction and A z in the z direction. So, we can write A to B i A x plus j A y plus k A z. Now, the displacement of any point on this body will make use of this function as usual u for the x direction, v for the y direction and w for the z direction.

This u; obviously, it is going to be a function of the coordinate of the point. So, they are function of... So, this u i if you try to write in the form of the tensor, it is going to be function of x, y, z, i it takes some value 1, 2, 3. And in specific in to the specific i equal to 1 indicates u and i equal to 2 indicates v and i equal to 3 indicates w. Now, we would like to consider that the displacement of the point P.

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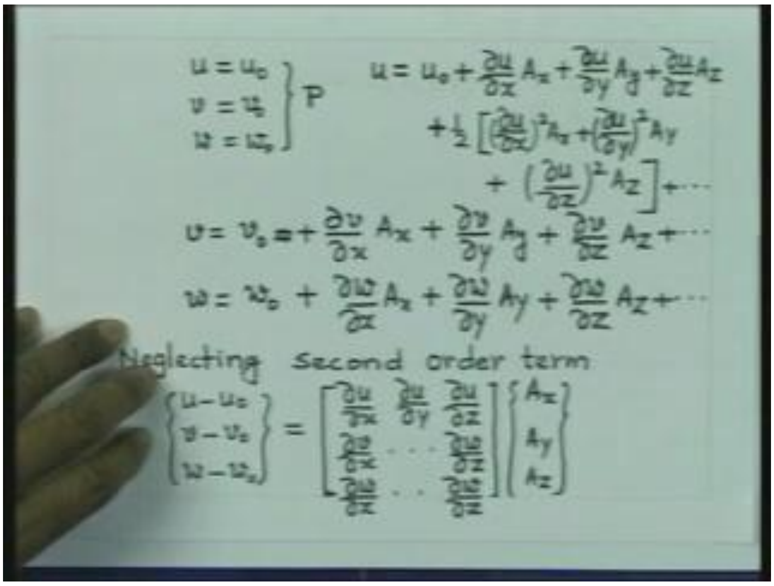

Let us say that displacement of the point P is u 0 and v is v 0, w is w 0. Now, you can write the displacement of the point Q, which is at a small distance apart from P. So, this are the displacements of the point P. Now, if you want to write the displacement of the point Q you can write like this u equal to u 0 plus you can write the ((Refer Time: 06:29)) expansion for the functions u i.

So therefore, we will have delta u delta x A x plus delta u delta y A y plus delta u delta z A z, that will be first order term. Then, we are also going to get second order terms which are nothing but delta u delta x square into A x plus delta u delta y square into A y plus delta u delta z square into A z and you can have higher order terms.

So, we can write similar expression for the other displacement. So, we can also be retained. So, v is nothing but v 0 plus delta v delta x A x plus delta v delta y A y plus delta v delta z A z plus we have the second order terms and third order terms And so on. Similarly, w can be written as w 0 plus delta w delta x A x delta w delta y A y plus delta w delta z A z plus the higher order terms.

Now, if we neglect the second order terms. We can write the following, you can write u minus u 0 v minus v 0 w minus w 0 this is nothing but a compact form. Then, we will have delta v delta x delta v delta y then delta w delta z. And the last row will have the derivatives of delta w delta x delta w delta y delta w delta z. So therefore, this multiplied by components of A x Ay A z, this can be written in an compact form using tensor rotation.

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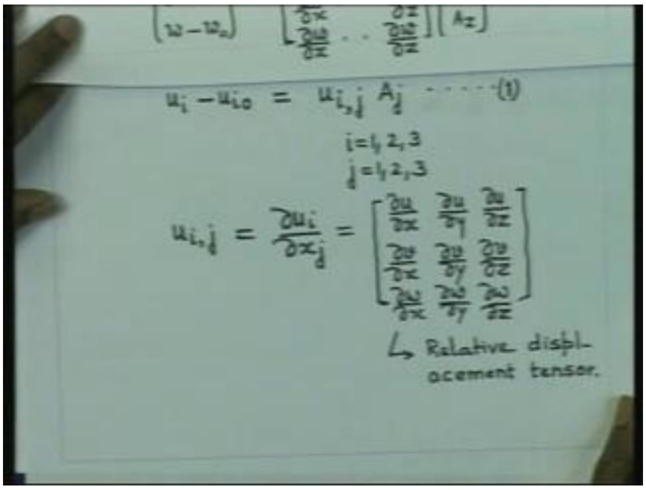

We can make use of that u i minus u i 0 is equal to u i comma j multiplied by A j. So, this relationship, let us indicate by 1. Here, in this gives the this u i minus u i 0 here in of course, you have assume that i is taking up value 1, 2, 3 and j is taking up value also 1, 2, 3. And this i comma j, this indicates really all the derivatives.

So, therefore, u i j in is nothing but delta u i delta x j. And here, we have all the 9 derivatives. And this is nothing but the this expanded form which we have given earlier. So, this consist of all the 9 derivatives. And this is known as relative displacement tensor. This, since the displacements between the two points P and Q are changing by ((Refer Time: 13:52)) this match, this indicates really the change in the length. So, p moves in the x direction by u 0 and Q moves by the distance u. So, therefore, in the x direction, there is a change in the length u minus u 0.

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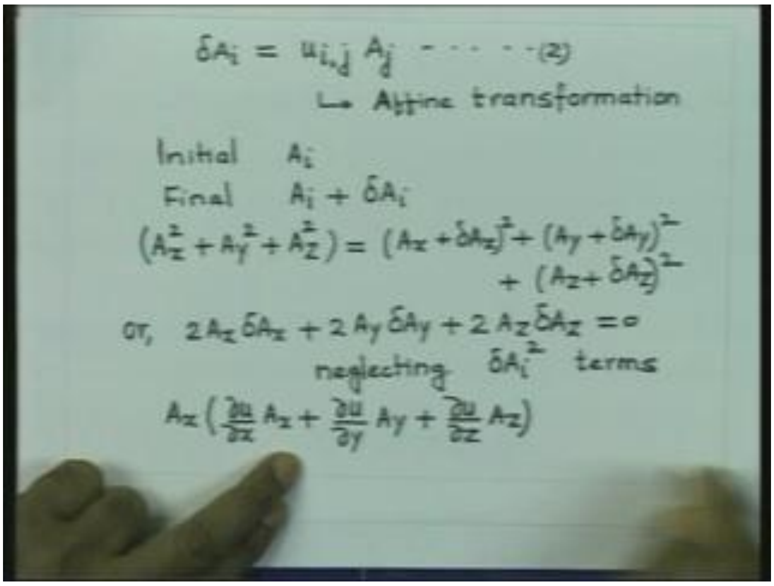

So, the change in the length of the distance P Q if we indicate by delta a i ((Refer Time: 15:40)). Then, we will have the components of the change are delta A x delta A y and delta A z. Then, it is very easy to realize, that this delta A i is nothing but u i minus u i 0. So, we can now write delta A i is equal to u i comma j multiplied by A j. This is known as affine transformation. Now, the changes in the length in the x, y and z direction between the two points is given by this affine transformation. So, the initial component of the distance are A i and the final components of the distance are A i plus delta A i.

Now, if you consider that the body displaced in such a way, that there was no deformation. That means, distance between P and Q remained unchanged, then we are going to get that the initial square of the length between the two points are given by A x square plus A y square plus A z square that will be initial distance square. And that must be equal to A x plus delta A x square A y plus delta A y square plus A z plus delta A z square.

Now, this will give us twice A x delta A x plus twice A y delta A y plus twice A z delta A z equal to 0 neglecting delta A i square terms. Now, we have the expression for delta x delta A y and delta A z. So, if we now consider and if we write then and it is going to give us A x into delta u delta x A x plus delta u delta y A y plus delta u delta z A z ((Refer Time: 18:57)). Plus A y into delta v delta A x delta v delta y A y plus delta v delta z A z.

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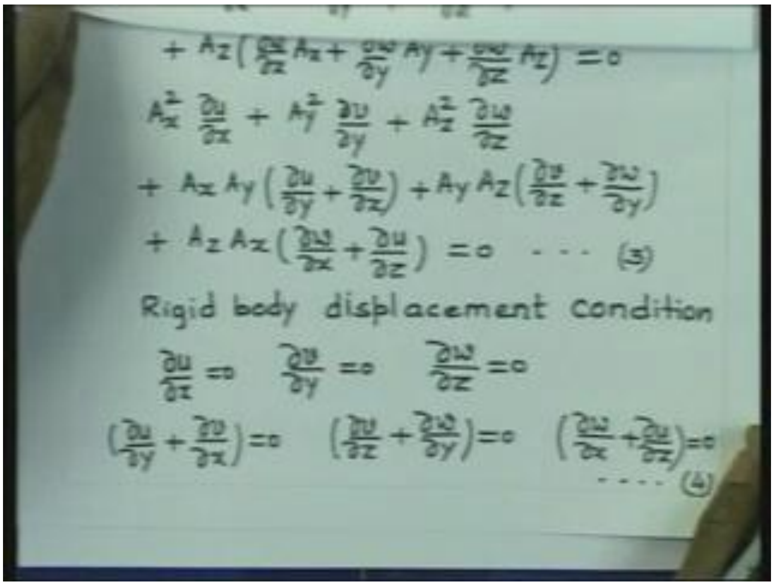

Plus we are going to get the third component plus A z into delta w delta x into A x plus delta w delta y into A y plus delta w delta z into A z equal to 0. So, this if we now rearrange it is going to be A x square delta u delta x plus A y square delta v delta y plus A z square delta w delta z. Plus you are going to get A x into A y delta u delta y plus delta v delta x plus A y A z delta v delta z plus delta w delta y.

Plus A z into A x delta w delta x plus delta u delta z equal to 0. Now, this relationship if we call it 3, if we have A x, A y, A z they are always positive and at the same time call this quantities are positive. So, therefore, if the left hand side is to be 0, this demand that the coefficient of all this terms have got to be 0. So therefore rigid body displacement condition.

So, the body is going to move like a rigid body without deformation is going to be coefficient of all these terms are 0. So therefore, it means that delta u delta x equal to 0 delta v delta y equal to 0 delta w delta z equal to 0. And then, we have delta u delta y plus delta v delta x equal to 0 delta v delta z plus delta w delta y equal to 0 and delta w delta x plus delta u delta z equal to 0. So, these six conditions are the conditions for the rigid body displacement.

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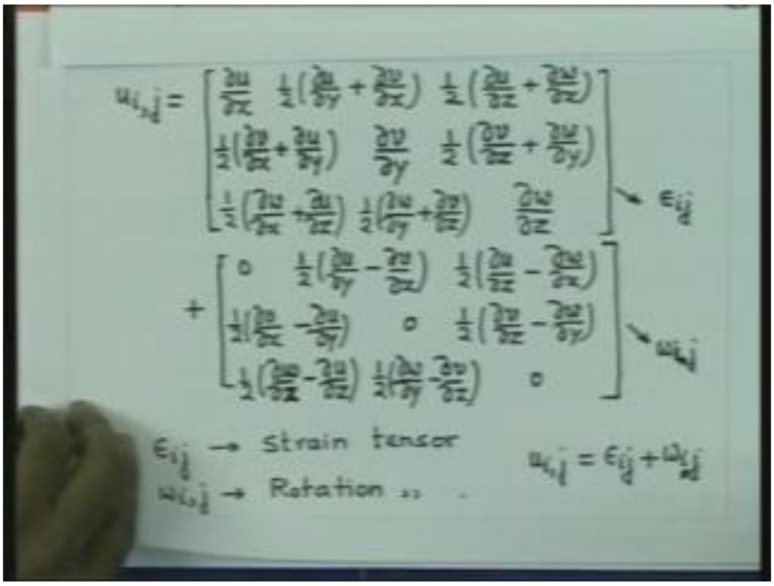

Now, the relative displacement tensor we can rearrange it, relative displacement tensor can be now rearrange like this, that u i j is equal to delta u delta x half delta u delta y plus delta v delta x half delta u delta z plus delta w delta x. Then, we have half delta v delta x minus delta u delta y delta v delta y half delta v delta z this is plus this is delta v delta z delta w delta y.

Then, we have half delta w delta x plus delta u delta z half delta w delta y plus delta v delta z and delta w delta z. So, this is the first path and then, we try to write plus 0, this is half delta u delta y minus delta v delta x half delta u delta z minus delta w delta x. Then, this is half delta v delta x minus delta u delta y 0 half delta v delta z minus delta w delta y.

Then this one half delta w delta x minus delta u delta z half delta w delta y minus delta v delta z 0. So, if we just write this we get back, you can see that the here we have added some thing we subtracted something. And then, this two terms gives me back the original expression for u i j. And you can see that, this is nothing but the strain tensor, this is the Epsilon x this is gamma x y, this is gamma x z, this is gamma y x Epsilon y gamma y z this is gamma z x, z y and Epsilon z and this is the strain tensor.

So therefore, this is the strain tensor Epsilon i j and this one is nothing but it is giving us the condition, that it is rotation this is the rotational tensor. So therefore, this gives the rotation tensor i j. So therefore, Epsilon i j is the strain tensor and w i j is the rotation tensors. So, you consider the relative displacement tensor is nothing but sum of the u i j relative displacement tensor is nothing but sum of the strain tensor and the rotation tensor here Epsilon i j we will indicate this thing, this will indicate this thing as w i j. So therefore, this is w i j. So, relative displacement tensor is sum of the strain tensor and the rotation sensor.

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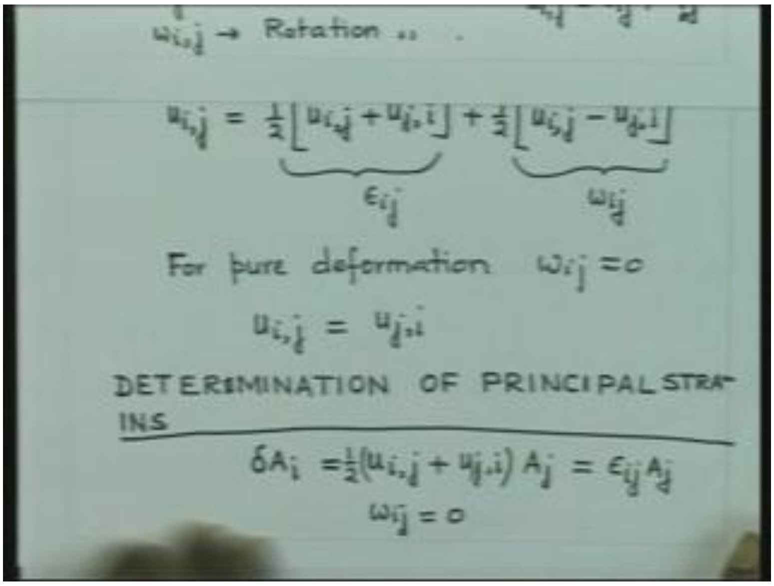

Now, this can be written also making use of the tensor relationship without detailing out, we can write it very in a compact form. U i j is nothing but half of u i j plus u j i plus half of u i j minus u j i. So, you get now the strain tensor, you are already familiar that this is the definition of the strain tensor. And this is the definition of the rotation tensor w i j. For pure deformation, pure deformation we will have no rotation for pure deformation w i j is equal to 0. And therefore, we have u i j is equal to u j i.

Now, you would like to consider having got this affine transformation which is now possible to consider determination of the principle strains. So, will now consider determination of principal strains. So, let us consider the determination of the principal strains. Now, for pure deformation we have delta A i is equal to u i j half of u i j u j i into A j. Or you can write this thing as Epsilon i j into A j. Since, w i j is equal to 0. Now, let us consider that this P Q that we consider is having it is aligned with the direction.

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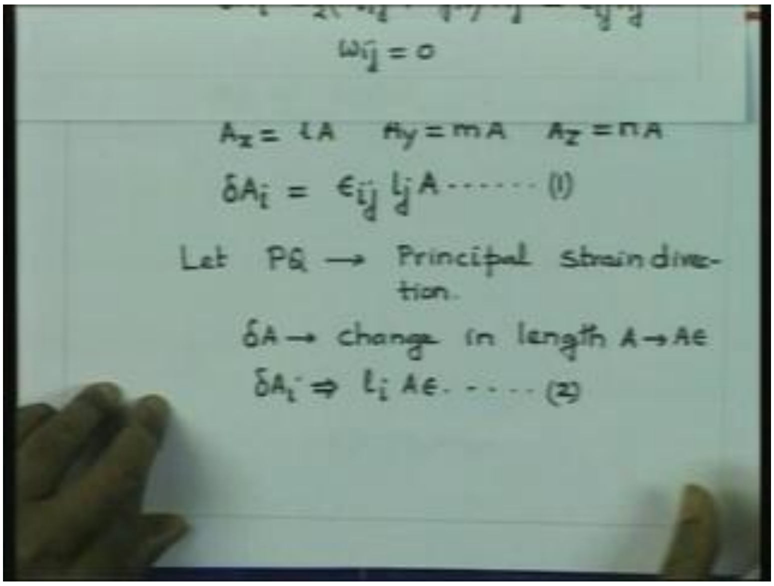

So, the P Q is aligned with the directions with cosines l, m and n. Then, in that case we can write that A x is equal to l into A, A y is nothing but m into A and A z is equal to n into A. Therefore, you can now write, we can write now that delta A i is equal to Epsilon i j l j into A. Now, if P Q happens to be the principal strain direction. Let P Q rate principal strain direction.

So, if P Q is the principal direction, then we can write that the changes in length delta A change in length of element P Q which is A is nothing but it is going to be A into Epsilon. So, we can now write that delta A i is nothing but it is we can write then delta A i to the l i into A into Epsilon. So, we have already got delta A i equal to Epsilon i j l j into A. And again we have then giving by this one. So, if you consider that this relationship, let us say it is 1, this relationship is 2.

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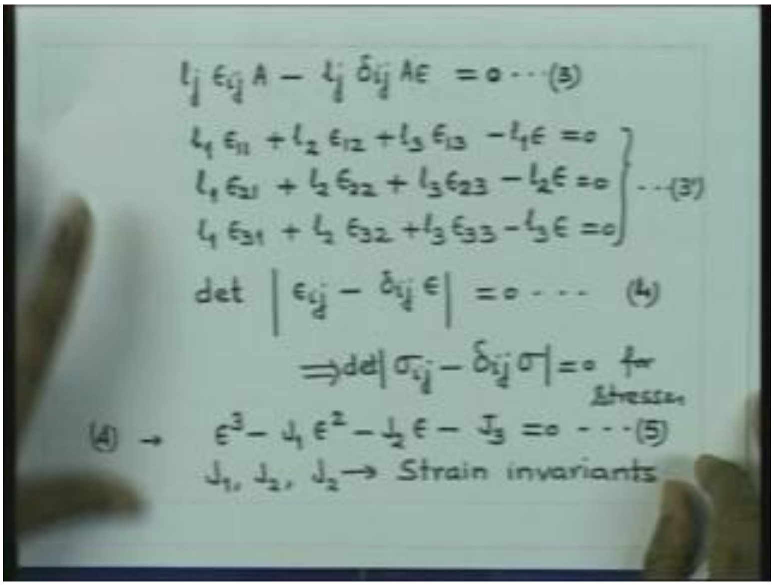

Now, you can combine this two relations and you will get now that l j Epsilon i j A minus l j delta i j A Epsilon is equal to 0. This Epsilon stands for principal strain and I must also indicate that this A is the length of P Q. So, this particular relationship, it is very similar to what we have obtain in the case of principal stress.

So, if you expand it, you will find that, this is nothing but l 1 into Epsilon 1 1 plus l 2 into Epsilon 1 2 plus l 3 Epsilon 1 3 minus l 1 into Epsilon equal to 0. And this is Epsilon l 1 into Epsilon 2 1 plus l 2 into Epsilon 2 2 plus l 3 Epsilon 2 3 minus l 2 into Epsilon equal to 0, And l 1 Epsilon 3 1 plus l 2 Epsilon 3 2 plus l 3 Epsilon 3 3 minus l 3 Epsilon 3 equal to 0. So, this is the expanded form of equation 3 let us indicate by 3 dash.

Now, this involves equation which can be written in a form like this, that the compact form is 3 and you say that here in we have the l 1, l 2, l 3 are the constant symbol, unknown constants. And this is the homogeneous set of equation right hand side is 0. And therefore, if it is to give us non zero solutions. Then, in that case we must have determinant of the quantity Epsilon i j minus delta i j Epsilon must be equal to 0.

So, this is the characteristic equation. And it is similar to if you remember, in the case of stresses we had determinant of sigma i j minus delta i j minus sigma equal to 0 for stresses for principal stresses. So, you have got a closely parallel relationship for the principal strains.

And this relation is going to again before can be expanded it is going to be of a form Epsilon cube minus j 1 into Epsilon square minus j 2 into Epsilon minus j 3 equal to 0. This is the cubic equation, involving the principal strain and this j 1, j 2, j 3 are called strain invariants, similar to stress invariants they are also strain invariants.

And they will be obtainable from the strain component. And they will have closely parallel relationship. And with this you can find out the three principal strains. And for each of the principal strains, you can come back to this relationship 3 dash or 3 to get the corresponding directions. So, that is how you can determine the principals strains and their directions.

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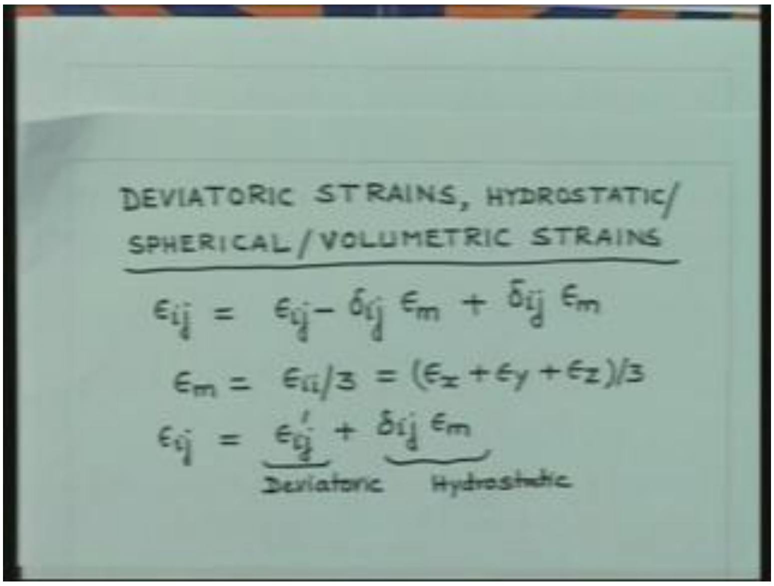

Similarly, we can also consider the deviatoric strains. So, deviatory strains in close parallel to deviatoric stress, we can have deviatoric strains. Then, we can have also hydrostatic strains, deviatoric strains and hydrostatic spherical volumetric strain. So, we would like to now break the total strains into deviatoric can volumetric strains.

And obviously, we can have Epsilon i j is equal to Epsilon i j minus delta i j into Epsilon m plus delta i j into Epsilon m. This Epsilon m is nothing but the mean stress mean normal stress and this is given by Epsilon i i by 3. And therefore, this is nothing but Epsilon x plus Epsilon y plus Epsilon z divided by 3.

And we can represent Epsilon i j in this form Epsilon i j dash plus delta i j Epsilon m varying, this is known as deviatoric strain tensor. And this one is known as volumetric or hydrostatic strain tensor or spherical strain tensor. So, we will write this thing as hydrostatic strain tensor. There are certain typical it is, which are associated with the strains. And we would like to look into the typical it is, this are known as strain compatibility conditions.

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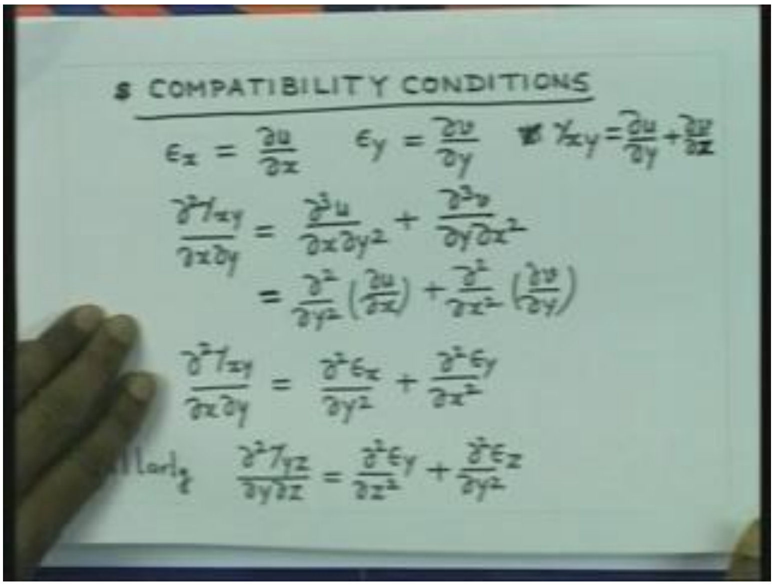

We had already got the expression for strains like Epsilon x delta u delta x Epsilon y equal to delta v delta y. And then, we have gamma x y is equal to delta u delta y plus delta v delta z. Now, you see that the three strain components are defined by the two displacement functions.

Now, if you try to take the second derivative of this, if you differentiate this with respect to x and y separately. Then, what will have is delta 2 gamma x y delta x delta y. So, we will have this thing as delta cube u delta x delta y square plus delta 3 v this would be x. So, this is delta y delta x square, now you can write this thing as delta 2 delta y square delta u delta x delta 2 delta x square delta v delta y.

So, we have now this is nothing but Epsilon x this is Epsilon y. So, we can write now delta 2 gamma x y delta x delta y is equal to delta 2 Epsilon x delta y square delta 2 delta x square delta 2 Epsilon y delta x square. So therefore, this is the compatibility condition, you see that here we have the derivative x y this is the ((Refer Time: 48:50)). And this is the related Epsilon x and the derivative is twice with respect to y and here it is Epsilon y and the derivative with respect to x.

So, this is known as the compatibility conditions strain compatibility conditions. If we now consider the other combination, we can have gamma y z and the strain Epsilon y Epsilon z by close parallel, we can get delta 2 gamma y z derivative should be with respect to y and z. This must be equal to delta 2 delta z square it must be Epsilon y delta 2 delta y square Epsilon z.

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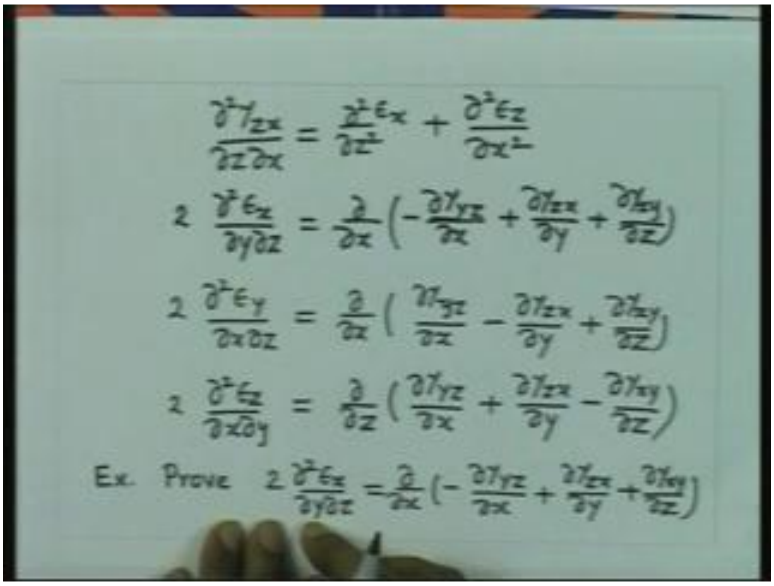

So, this is the second strain compatibility conditions, in three dimension and the third one we can write now delta 2 gamma z x delta z delta x is equal to delta 2 delta z square Epsilon x plus delta 2 Epsilon z delta x square. So, this three relations are known as three compatibility conditions in three dimensions. And if we have two dimensions, we are going to simply have this one strain compatibility conditions.

It can be also shown that we will have another set of compatibility conditions. We will consider now, this relationships are delta 2 Epsilon x delta y delta z is equal to delta delta x minus delta gamma y z delta x plus delta gamma z x delta y plus delta gamma x y delta z. So, this is another strain compatibility conditions.

And the other two which are of the similar type delta 2 Epsilon y delta x delta z delta delta x delta gamma y z delta x minus delta gamma z x delta y plus delta gamma x y delta z. And the last one delta 2 Epsilon z delta x delta y is equal to delta delta z delta gamma y z delta x plus delta gamma z x delta y minus delta gamma x y delta z. This compatibility conditions can also be derived starting from the strains.

So, what we find is that we have the strains satisfying some constants conditions, they are three in this form and the other three are going to be in this form. Now, it will be good idea for you to see, how this condition can be derived. Now, if you try to consider... So, let us take as an example how to derive the compatibility conditions, which is shown here.

So therefore, prove delta z delta delta x minus delta gamma y z delta x delta gamma z x delta y plus delta gamma x y by delta z. You not of to do this, what you have to do? You just take the expression for delta gamma x y differentiate with respect to z. And then, take the expression for delta z x differentiate with respect to y. And similarly take the expression for gamma delta gamma y z and differentiate with respect to x.

And then, take the sum and once you take that sum, you will find that you are going to have a relationship involving some expression, which will finally, lead to this left hand side. So, you try to take the expression for the strains gamma x y gamma z x gamma y z, separately. And differentiate them as shown here some them up. Then, once you differentiate this resultant expression with respect to x, you are going to get the left hand side.