Advanced Strength of Materials
Prof. S.K. Maiti
Department of Mechanical Engineering
Indian Institute of Technology, Bombay

Lecture – 35

So, we will continue our discussion on bending of circular plates subjected to axis symmetric loading. We gave already seen certain points. So, let us just look at them once again. We have got the equation of the deformation in this form, d 2 phi dx square plus 1 by x d phi dx minus phi by x square is equal to minus V by D. Wherein this phi is nothing but slope and this slope is related to the deflection by this relationship, where the slope is negative. So, that is why it has become in that form.

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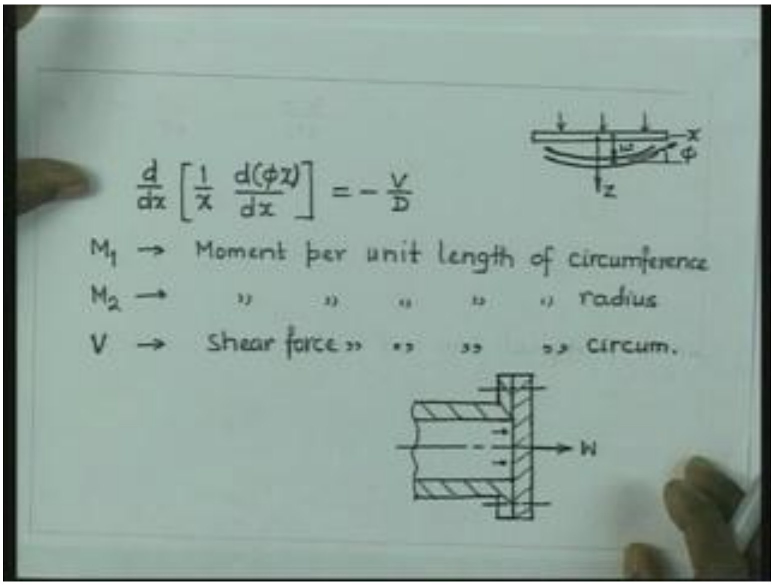

So, let us just try to again sketch a portion of the plate. So, this is the plate and this is our x axis and starting from the centre, we have the z axis down and another action of the loading like this. We are going to see, that the formation of the plate in this manner and therefore, the slope is now going to be at any point, the slope is like this. That is what is phi and, then phi is equal to minus dw dx, because you consider w to be this distance. That is the distance of the deflection of the centre from the original position.

Now, this equation, it is good to work with this form 2. And you have come across, this type of equation in connection with the thick cylinder. We have similar equation and it can be written in a form. So, d by dx 1 by x d phi x dx is equal to minus V by D. And some of the variables that we introduced, it was M 1. M 1 was moment per unit length of circumference.

Similarly, M 2 was moment per unit length of radius and V was shear force per unit length of circumference. Let us now consider some specific examples. Think of a pressure vessel. So, think of this pressure vessel, with the enclosure which is circular and they could be facing through each other by bolts, they could be there could be welding as well. So, this type of, this portion of the plate is subjected to pressure loading like this. We can have some external force also acting. May be some load coming up on this plate like this.

So, this is a case of loading wherein, we can now try to look into what sort of deformation. This particular enclosure will undergo and what sort of stresses are going to come up. Now, the end conditions that are going to be persisting here; so if, you would like to consider, that the plate is up to this much, then the end condition at this ends. They can be considered to be clamped or they can be also considered as simply supported. So, we will like to consider both the cases. So, the conditions that can exist at this point and at this point, it can conform to plant condition or simply supported conditions. So, we will now try to consider a general case.

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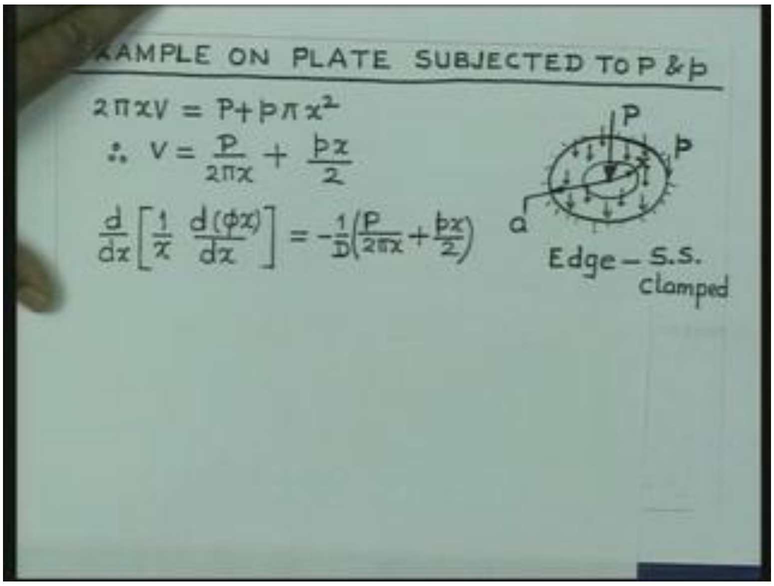

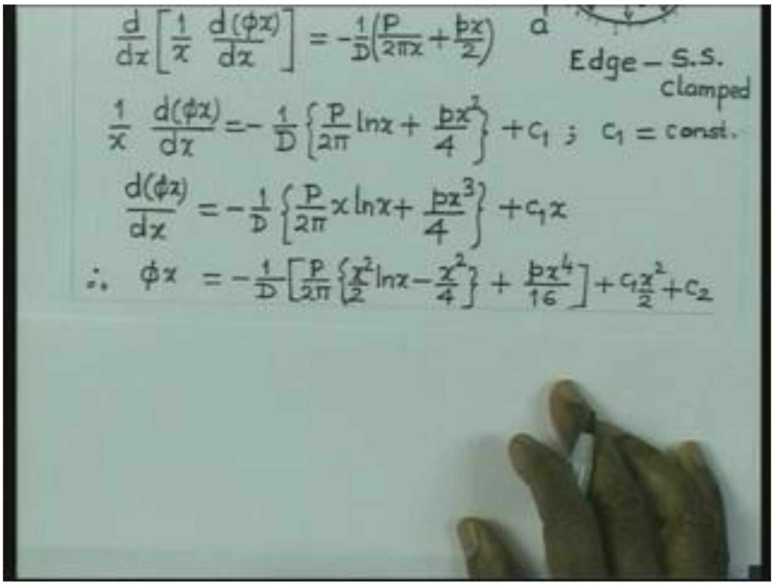

So, we have a situation like this, that the plate is let us say, it is supported at the edges and it could be simply supported or plant. So, it could be plant as well. So, let us write both the possibilities and we will just go ahead as a general derivation. So, the loading is on the plate, some normal load plus distributed load acting. Now, in order to find out the solution, we must know, what is the shear force per unit length, of circumference at a particular position.

So, let us now consider a position which is at a radial distance of x. So, therefore, let us consider a circle. So, this is the circle of radius x and if, we now segregate this portion, we will have forces shear forces acting on its boundary and we can now write; if, we consider that V is the shear force per unit length on this boundary. Then, in that case we can write now 2 pi x into V which is going to act in the upward direction and that is to be counter balanced by P plus p into pi x square.

So, that is the force balance and this force balance, is going to give us the value of the shear force per unit length that we need. And therefore, V is equal to P by 2 pi x plus p into x by 2. That's the shear force. We can substitute this value, into the equation of governing equation of deformation of a circular plate d by dx 1 by x d phi x dx is equal to minus 1 by D, for V we write P by 2 pi x plus px by 2.

Now, we this is easily integrable. So, if you do that; 1 by x d phi dx minus 1 by D P by 2 pi lnx plus px square by 4 plus, there will be 1 arbitrary constant of integration, so that is c 1, let us say, so this is a constant. Again integrating once more, we get or rather we can rewrite this expression in this form; P by 2 pi x lnx plus px cube by 4 plus c 1 into x. and if we integrate once more, this gives us phi x is equal to minus. This is the case of product of 2 functions. So, you can integrate this first that will give us x square by 2 lnx. Then, minus x square by 2, differentiation of this will give us 1 by x. So, finally, we will have x by 2 and integration of that will give us x square by 4. So, we are going to get now; P by 2 pi x square by 2 lnx minus x square by 4 plus px 4 by 16 plus c 1 x square by 2 plus another constant of integration will come.

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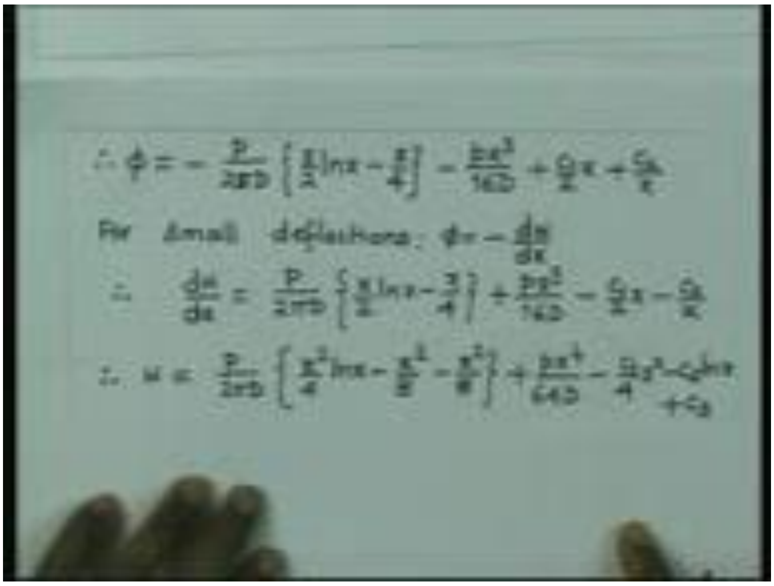

Let us say that is c 2. So, if we now write finally, phi. phi is P by 2 pi D. We have x by 2 lnx minus x by 4 minus px cube by 16 D plus c 1 by 2 x plus c 2 by x. So, we have got now the expression for phi the slope and now we can certainly go for finding out the deflection since, phi is related to the deflection by the formula phi is equal to minus delta w delta x.

So, we can now consider phi as delta w minus delta w delta x. And therefore, this is of course, for small deformation because for small deformation, we can write tan phi is equal to phi and therefore, small deflections we can write that. And hence, we have dw dx is equal to P by 2 pi D x by 2 lnx minus x by 4 plus px cube by 16D minus c 1 by 2 x minus c 2 by x.

Therefore, W deflection is obtained on integration which is nothing but P by 2 pi D and integration of this is again a product of 2 functions. So, we will now get x square by 4 lnx minus x square by 8. From this we get this 1 and this 1 will give us x square by 8. This will give us p x 4 by 64D minus c 1 by 4 x square minus c 2 lnx plus c 3 wherein again, c 3 is a constant of integration.

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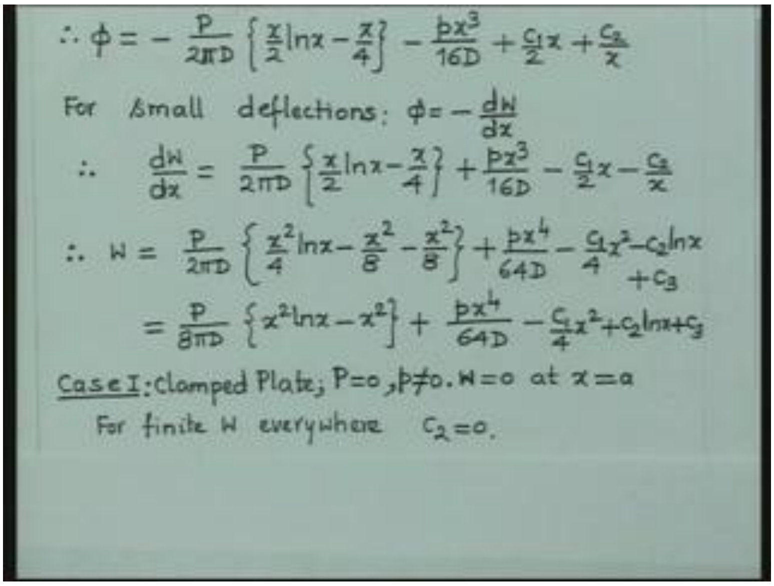

So, if we now combine the similar terms, we have P by 8 pi D x square lnx minus x square px 4 by 64D minus c 1 by 4 x square c 2 lnx plus c 3. This is the complete expression for W in the case of loading, where we have concentrated load P and distributed expression p. Now, let us try to go for specific problem where the plate is clamped and let us also consider that P is absent. There are no concentrated pores, but there is uniformly distributed pressure over the whole plate.

So, let us try to consider again, the end conditions to be clamped fully clamped plate. So, we have now a case, here it is clamped. Further, we have P equal to 0 and p is not equal to 0. You can see that the boundary conditions are obvious, will have deflection equal to 0. If the dimension of the plate is considered to be a, then at x equal to a w equal to 0. Now, the expression for w that we have here it means: that if you have, if you try to calculate the deflection at the point x equal to 0 that is, at the centre of the plate, this term will shift to infinity, but we cannot have infinity deflection at any point on the plate. So, for finite deflection everywhere, we must have this constant c 2 equal to 0.

So, for finite W everywhere, c 2 equal to 0. That is 1 constant is resolved. Now, we will make use of the other boundary conditions. To find out the constants, other boundary conditions happens to be are as follows. We have the slope at the end of the plate x equal to a, that is 0.

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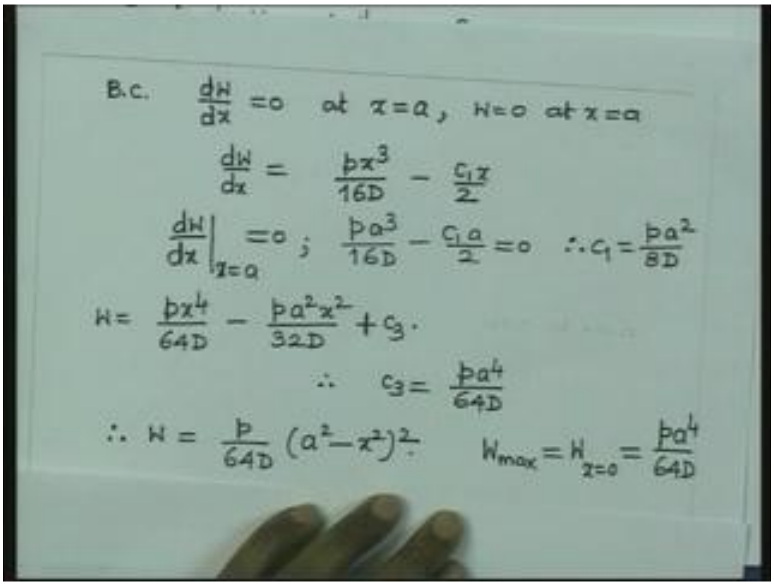

So, we can write now boundary condition, that the slope dW dx equal to 0 at x is equal to a and at the same time, you do not have any deflection at x equal to a. So, therefore, another boundary condition that we have already stated earlier, is that W equal to 0 at x equal to a. So, these 2 conditions are going to suffice, to find out the 2 constants c 1 and c 3. So, let us now write dW dx, dW dx we have only this part of the expression because this P is 0.

So, therefore, it is going to be px cube that 4 will cancel and we will have this minus c 1 x by 2. Using the first boundary condition, dW dx at x equal to a at 0, we have pa cube by 16D minus c 1 a by 2 equal to 0. So, therefore, constant c 1 is equal to pa square by 8D. That's the constant first constant. Now, we can write therefore, W at px to the power four by 64D minus pa square x square 32 by 32D plus c 3. That's the expression for W and now W equal to 0 at x is equal to a.

So, from that if you put x is equal to a then ,in that case we are going to get c 3 to be equal to pa 4 by 64D minus pa 4 by 32D and that will give us c 3 equal to pa 4 by 64D. Finally, we have got all the constants and now when you put this in this expression, it forms a quadratic. Therefore, we have W equal to p by 64D a square minus x square whole square. So, that is deflection of the plate.

Now, we have to calculate, we are in a position to calculate the deflection very easily. It is obvious; that at x equal to 0 we are going to have the maximum deflection and therefore, W maximum, we can write W maximum and that occurs at x is equal to 0 and which is nothing but pa 4 by 64D. Now, we would like to calculate our other interest, to calculate the maximum stresses. Deflection is resolved. So, therefore, let us now calculate the maximum stresses. For that we must be able to calculate, the maximum bending moment M 1 and M 2 at the critical location.

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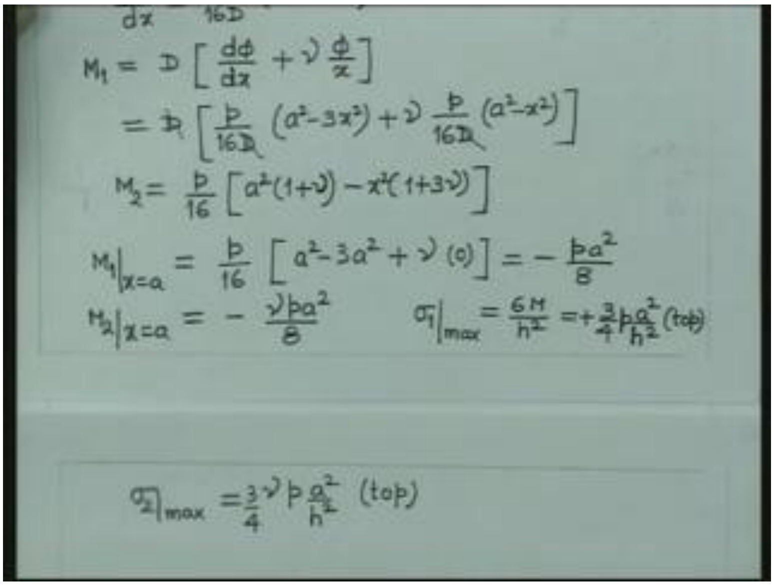

So, for doing that we will have to find out the radius of curvature and therefore, now we will write phi as minus dW dx and that is equal to minus p by 64D 2 a square. So, we are just making use of this relationship and we have this multiplied by 2x and that is nothing but p 16D x into a square minus x square. So, that is the expression for the slope phi and we can now write, d phi dx is equal to p by 16D a square minus 3x square. So, you can write this thing as a square x minus x cube. So, that gives us d phi dx.

We have the bending moment M 1 which is in the x z plane D into d phi dx plus nu phi by x. So, if we substitute the values p by 16D a square minus 3x square plus nu phi by x. So, therefore, it is p by 16D minus a square minus x square. So, D get cancelled. This D will cancel everywhere. So, it will become independent of D. So, we will have M 1 equal to p by 16 a square minus 3x square plus nu into p by 16 into a square minus square.

Similarly, M 2 is going to be given by D into phi by x plus nu times d phi dx. So, we can now write M 2 which will be like this; p by 16 a square 1 plus nu minus x square 1 plus 3 nu. So, M 1 at x equal to a. We see that the, at x equal to a we are going to have the maximum bending moment. We will try to calculate the bending moment at x equal to a. So, M 1 at x is equal to a is equal to p by 16 a square minus 3a square nu into 0 and therefore, this is nothing but pa square by 8.

Similarly, M 2 at x is equal to a is equal to, here x is equal to a. So, we will have this term canceling, here it is nu and here it is 3 nu. So, we will be left with 2 and 2 will cancel with 16. So, therefore, we will have nu pa square by 8. So, these are the bending moments at the end of the plate. Now, let us try to see how the stresses are going to come up. Because of this bending moment, we will have a stress which is given by extreme fiber stress, which is maximum which is nothing but can be calculated from the formula 6M by h square, because here p equal to unity. So, 6M by h square p is unity.

So, we can write now 6M by h square and since, this is pa square by 8. So, we can now have this substituted here and this will give us 3 by 4 pa square by h square. So, this is the stress. It is if, we consider the stress at the top fiber that is the value top. Likewise if, we calculate the maximum stress due to M 2, sigma 2 maximum is equal to it is simply nu pa square by h square. So, here in the value is this and therefore, we will have nu pa square 6 times this divided by h square. So, that gives us we will again have of course, 3 by 4 factor will be there and this is the stress at the top fiber. It is very important to see, how this both the stresses are going to be positive at the top fiber.

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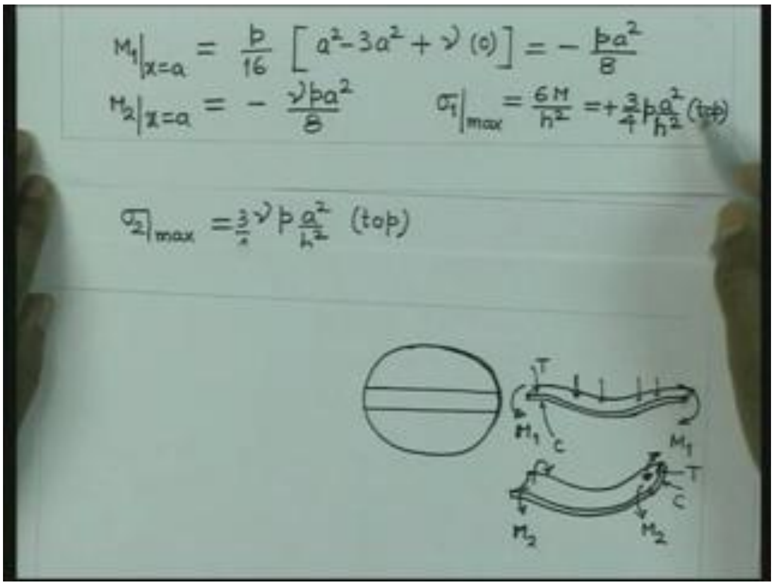

So, let us look into, look at the plate which is circular. Let us now consider only a portion of it at the centre and I have drawn it here. So, this is the portion which is subjected to bending. So, because of the pressure acting all over, it is going to bend like this and you see here, that the curvature that is going to be here is negative, whereas the curvature here that is going to come up is positive curvature and this is possible as the bending moment is acting like this. Already we have seen that M 1 and M 2 both are negative bending moment.

So, this M 1 at the end is minus pa square by 8. So, its negative bending moment and it acts like this. Now, under the action of this bending moment, we will have tensile stresses developed at the top, compressive stresses at the bottom. So, therefore, you here expect tensile stresses and at the bottom, we are going to get compressive stresses. And anticlastic curvature will develop such that, since compression is there, there will be extension here and contraction at the top. So, the anticlastic curvature will develop which is going to look like this. The plate is going to really, it is going to deform at the edges like this.

Since, it is clamped, this sort of curvature cannot come up and therefore, we will find that, there will be bending moment developed like this which is M 2. So, this bending moment will try to, this bending moment will try to develop like this to undo that curvature. So, therefore, this bending moment is negative, because you have considered the other direction to be positive. So, therefore, this is also negative bending moment. That's what have come and under the action of this type of bending moment we can; obviously, see that at this point, at the bottom we are going to see compression. And at the top we are going to see tension.

So, therefore, we will have tension at the top compression at the bottom. So, in both the cases, you see that we are going to have tension at the top and compression at the bottom. And these stresses are tension at the top compression at the bottom that is why, these stresses are both positive.

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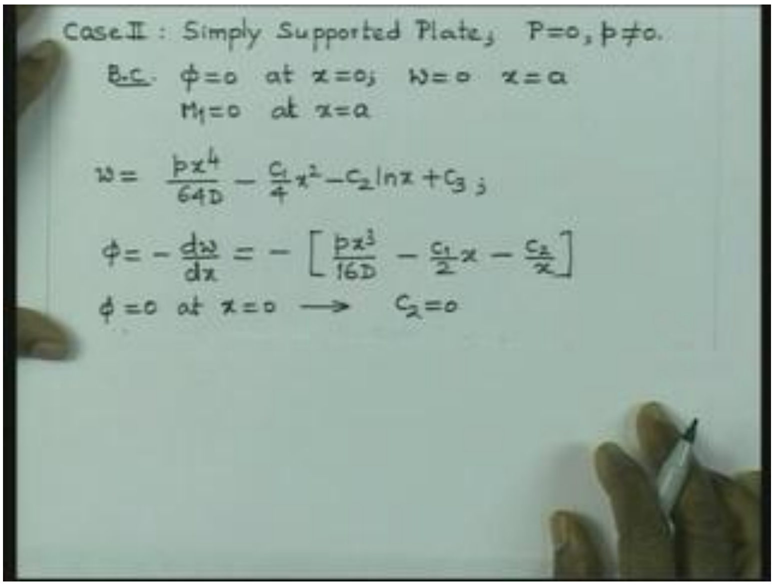

Now, we will try to consider the case of a simply supported plate. Let us see the other extreme, when a plate is simply supported what happens. Again we are trying to look for the case, when P is 0 and p is not equal to 0 and in the case of the simply supported plate, we would have conditions like; at x is equal to a, at the outer edge of the plate we will have deflection equal to 0 and bending moment equal to 0. A bending moment in the circumferential direction will be 0.

So, M 1 equal to 0 and we are going to get the slope at the centre of the plate to be 0. So, we can now write the boundary conditions phi equal to 0 at x equal to 0. Then, W equal to 0 at x is equal to a and bending moment equal to a. Bending moment of course, M 1 equal to 0 at x is equal to a. And our case is P is 0 and p is not equal to 0. We proceed as before. In this case we start with this expression of w w is equal to px 4 64D minus c 1 by 4 x square minus c 2 lnx plus c 3.

Now, we can write phi minus dw dx is equal to minus px cube by 16D minus c 1 by 2 x minus c 2 by x. The slope as we have slope equal to 0 at x equal to 0. So, phi equal to 0 at x is equal to 0, and that means this term is going to shift to infinity. Therefore, this constant must be equal to 0 to have finite value of the slope.

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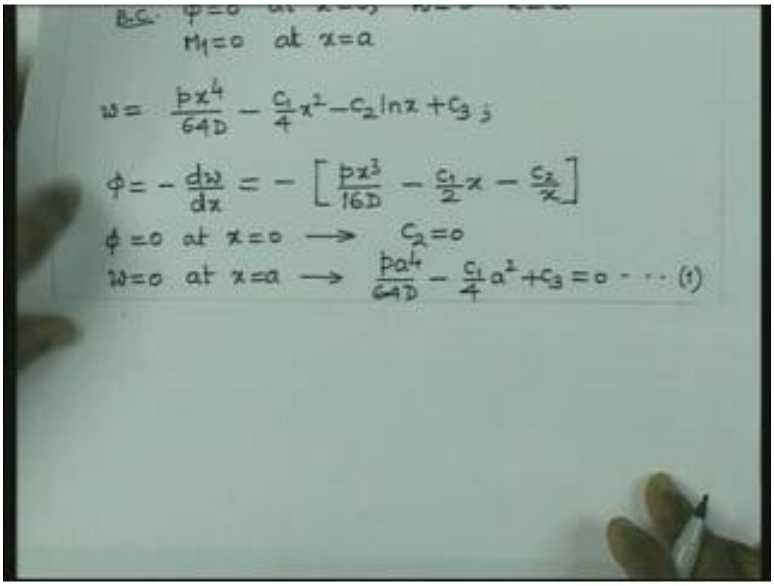

Finally therefore, we can now make use of the condition that w equal to 0 at x equal to a. So, we have now we will just substitute w equal to 0 at x is equal to a, noting that we have already got c 2. So, therefore, this is going to be pa 4 64D minus c 1 by 4 a square plus c 3 equal to 0. So, this is equation number 1 we have got 2 constants involved.

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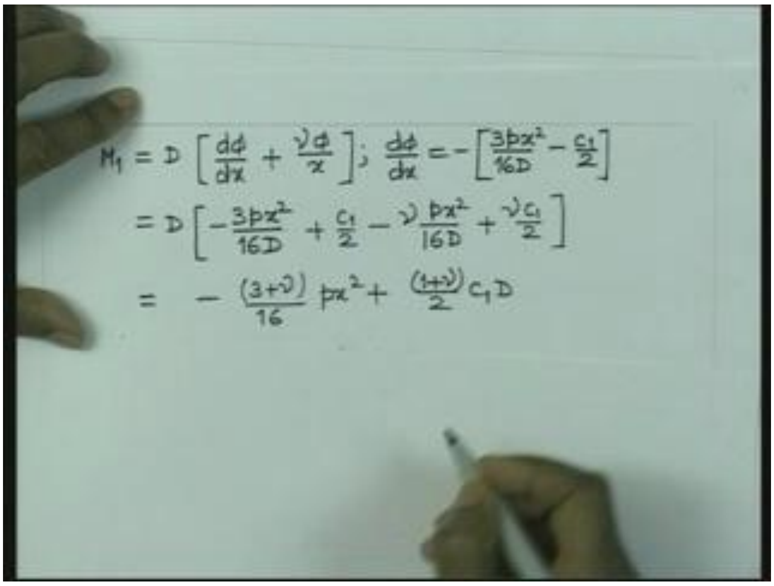

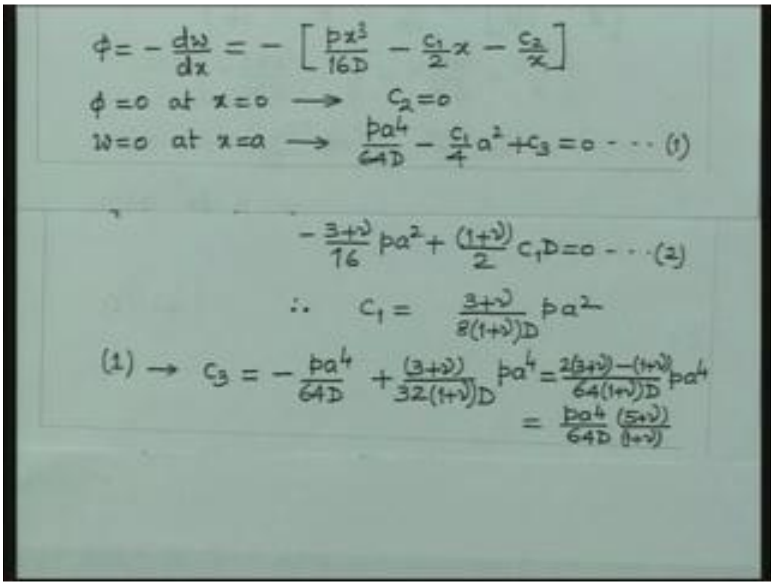

Now, we can make use of the other boundary condition provided we have the expression for bending moment M 1. So, if we consider bending moment M 1 which is, D into d phi dx plus nu times phi by x and therefore, d phi dx is equal to minus 3px square by 16D minus c 1 by 2. And hence we can now write D.M 1 as D into 3px square by 16D plus c 1 by 2 minus nu times phi which is phi by x of course, we have p x square by 16D nu c 1 by 2. That's the value for M 1 and since M 1 is equal to 0 at x is equal to a. So, this of course, we can write in a neater form, because this D is going to cancel for this 2 terms.

Therefore, we can write now 3 plus nu by 16 p x square plus 1 plus nu by 2 c 1 into D and since, M 1 equal to 0 at x is equal to a, we now get equation involving c 1 like this and therefore, we get c 1 equal to 3 plus nu by 8 into 1 plus nu into D pa square. And if we get back to earlier equation 1, which is here; from that equation, substituting the value of c 1 we can now obtain c 3. So, therefore, c 3 is equal to minus pa 4 by 64D plus a square by 4 c 1. So, therefore, it will be 3 plus nu by 32 into 1 plus nu into D pa 4 and that gives us 2 into 3 plus nu minus 1 plus nu by 64 into 1 plus nu D pa to the power 4, which simplifies to pa 4 64D phi plus nu by 1 plus nu. So, we can now substitute the value of c 1 and c 2 and recover the expression for deflection and slope.

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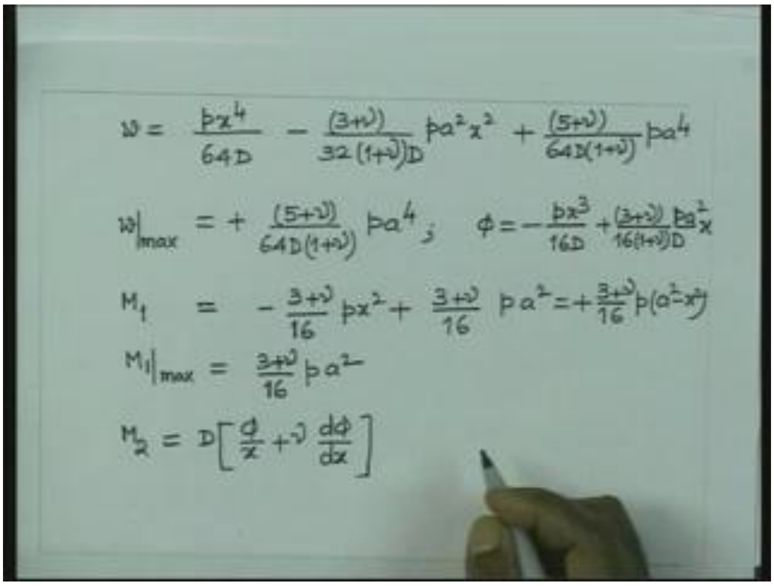

So, let us now again put the values back and get the expression for w w is equal to px four by 64D minus 3 plus nu 32 into 1 plus nu D pa square x square plus phi plus nu by 64D into 1 plus nu p raise to the power 4. And you can see that, the maximum deflection of the plate is going to occur at x equal to 0 and the value of the maximum deflection is given by this expression.

So, in the case, the maximum deflection is going to occur again at the centre and this maximum deflection is equal to phi plus nu 64D 1 plus nu pa raise to the power 4 and the value of phi is now px cube by 64D plus 3 plus nu 16 1 plus nu into D pa square into x. That’s the value for phi and if, we calculate M 1 which is, M 1 is nothing but D into d phi dx plus nu phi by x. So, we have everything.

So, if we substitute, the expression for M 1 turns out to be M 1 is equal to 3 plus nu by 16 px square plus 3 plus nu by 16 pa square. So, this as a simplified form 3 plus nu by sixteen pa square minus x square and hence, value of maximum bending moment is equal to 3 plus 16 3 plus nu 3 plus nu by 16 pa square.

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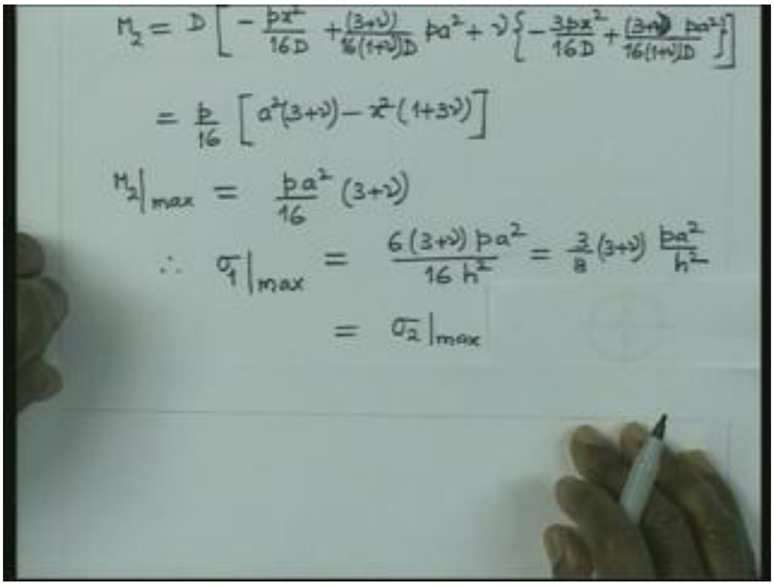

Similarly, if we calculate M 2 which is nothing but D into phi by x plus nu times d phi dx. So, D phi by x plus nu d phi dx, one should do the substitution in this, we get M 2 minus px square 16D plus 3 plus nu 16 1 plus nu D pa square plus nu times d phi dx 3 plus nu by 16 1 plus nu D pa square. So, it is 3 plus nu by 16 into 1 plus nu pa square by 16 1 plus nu into D. And this gets simplified; a square 3 plus nu x square 1 plus 3 nu this expression. And hence, M 2 maximum is at x equal to 0 which is pa square by 16 into 3 plus nu. So, finally, we have been able to get the maximum of both the moments

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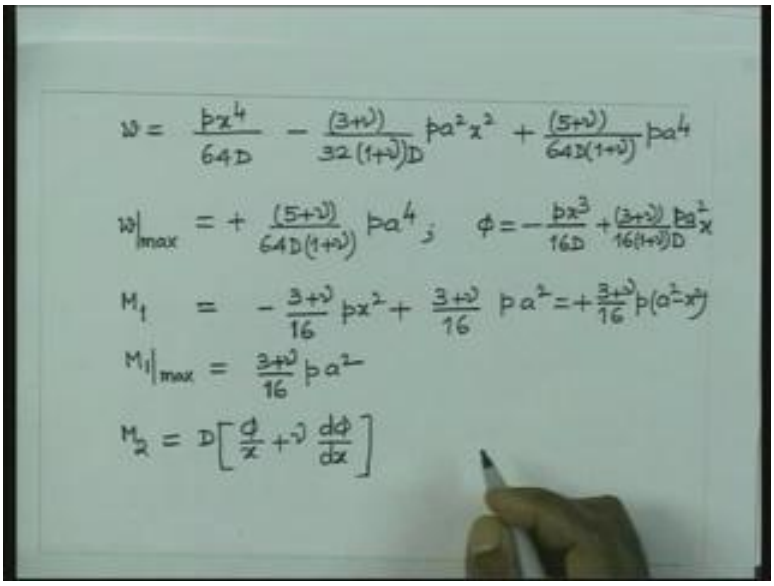

M 1 maximum we have got already 3 plus nu by 16 pa square. M 2 maximum is also of the same magnitude and hence now, we have at x equal to 0 sigma 1 max is equal to, it is nothing but 6 times the moment divided by h square and therefore, that will become, 3 plus nu pa square by sixteen h square and this is 3 by 8 3 plus nu pa square by h square. And in fact, the stress in the circumferential direction is also the same. So, therefore, sigma 2 maximum is also of the same magnitude. You can think physically also to establish that sigma 1 maximum is equal to sigma 2 maximum.

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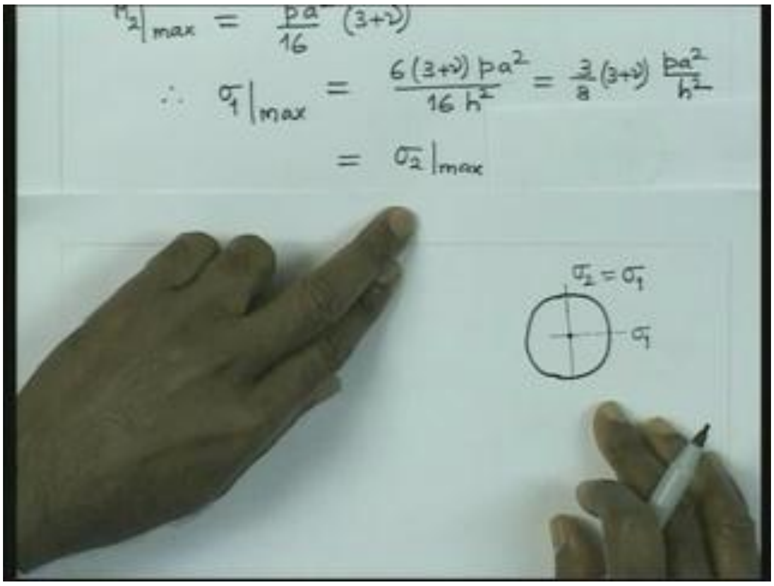

We can consider the plate. This is the centre of the plate and we have seen that, at the centre sigma 1 max is equal to 3 by 8 3 plus nu into pa square by h square. If you consider that, this is the direction of sigma 1 which is a radial direction. Now, the perpendicular direction is this 1. This is also a radial direction. Since, this is axis symmetric, the stress in this direction which we indicate to be sigma 2, it has got to be equal to sigma 1 and hence, at the centre the stress in this direction is maximum. So, also it is going to be maximum in this direction and hence sigma 1 maximum is equal to sigma 2 maximum.

The picture is not different in the case of clamp plate as well. Only the boundary condition will change, but at the centre we will have again sigma 1 maximum is equal to sigma 2 maximum. For the case of simply supported plate and also for the case of clamp plate, we can find out the stress variation along the radius and they can be compared. We will compare this in our next lecture.