Matrix Theory
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Lecture 50
Properties of Convergent Matrices

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The last time we discussed about non derogatory matrices and how one can determine commutativity of two matrices when they are non derogatory. So, specifically, non derogatory matrix is one for which the geometric multiplicity of every eigenvalue equals 1. And we proved one theorem which said that if A is a non derogatory matrix, then B commutes with A if and only if there exists a polynomial p of degree at most n minus 1 such that B equals p of A.

Also just made a small remark that the converse is also true that is, A is non derogatory if and only if every matrix that commutes with A is a polynomial in A. So, one quick question for you, if I take the identity matrix, is it non derogatory?

Student: No, Sir

Professor Chandra R. Murthy: No, it is not non derogatory, obviously, because the identity matrix it commutes with every other matrix, but it cannot be written any other matrix cannot be written as a polynomial of the identity matrix, any polynomial of the identity matrix will end up being a scaled version of the identity matrix. So, you cannot in general write some matrix that

commutes with A as a polynomial of A where A is the identity matrix. And so, the identity matrix is not derogatory, is not non derogatory. How about the all-ones matrix?

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T_2 : $A \in \mathbb{C}^{n \times n}$ nonderogatory. Then, $B \in \mathbb{C}^{n \times n}$ commutes w/ A iff $\exists$ a poly. $p(\cdot)$ of degree at most $(n-1)$ s.t. $B = $ [blue/red icons]

Remark : $A \in \mathbb{C}^{n \times n}$ is nonderogatory iff every matrix that commutes with A is a polynomial in A.

Today : Convergent matrices.
         Polynomials and matrices.

Recall : $A \in \mathbb{C}^{n \times n}$ convergent if all elements of $A^m \to 0$ as $m \to \infty$.

Diagonal : A convergent iff (all EVals (A)| < 1.
Extends directly to diagonalizable matrices.

Extension to nondiagonalizable matrices : Use JCF!

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Extension to nondiagonalizable matrices : Use JCF!

$A = SJS^{-1}, \quad A^m = SJ^m S^{-1}$

$A^m \to 0$ as $m \to \infty$, iff $J^m \to 0$ as $m \to \infty$.

$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 \\ & \ddots & \ddots \\ 0 & & \lambda \end{bmatrix} = \lambda I + \begin{bmatrix} 0 & 1 & 0 \\ & \ddots & \ddots \\ 0 & & 0 \end{bmatrix} = \lambda I + N_k$

$N_k = J_k(0).$ $N_k^m = 0 \quad \forall \; m \geq k.$

$[J_k(\lambda)]^m = (\lambda I + N_k)^m = \sum_{i=0}^{m} \binom{m}{i} \lambda^i N_k^{m-i}$

So, today we will discuss a little bit about convergent matrices. Now, recall that a matrix is convergent if all the elements of A power m go to 0 as m goes to infinity. So, we also know that a diagonal matrix is convergent if the magnitude of all the diagonal elements of that matrix are less than 1. And the diagonal entries are the eigenvalues of the matrix. And so, you can say more generally that if the magnitude of all the eigenvalues is less than 1 then the matrix is convergent. This result extends directly to diagonalizable matrices also and we have seen that before.

Now, using the Jordan canonical form, we can extend this idea to non diagonalizable matrices as well. So, if A equals SJS inverse, then A power m equals S J power m S inverse. So, if A power m goes to 0 as m goes to infinity, then this is true if and only if J power m goes to 0 as m goes to infinity. So, the question is when does J power m go to 0? So, now J is a block diagonal matrix, so, it suffices to consider the behavior of a single Jordan block because each of the blocks get raised to the power m when you raise the block diagonal matrix to the power m.

So, if I consider that single Jordan block, say Jk of lambda, which is basically a matrix with lambdas on the diagonal, and once on the first super diagonal and 0s everywhere else, and this can be written as lambda times the identity matrix plus this nilpotent matrix, which has 0s on the diagonal and 0s below here and 1s in the first super diagonal 0s everywhere else. And so, we will call this lambda i plus Nk.

This is the k cross k matrix which when raised to the power k will give you the all 0 matrix. So, specifically Nk is equal to Jk of 0. Now, Nk power m equals 0 for every m greater than or equal

to k. So, that means that if I take Jk of lambda power m then this is equal to lambda i plus and Nk power m which can be written as the summation i equal to 0 to m, m choose i lambda power i times Nk power m minus i and using this property here, we can simplify this as summation i equal to m minus k plus 1 to m, m choose i lambda power i Nk power m minus i other times will be equal to 0 because Nk to that power goes to 0 and this is true for all m greater than or equal to k.

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Now, so, the diagonal elements which is the i equal to mth term are all lambda power m lambda power m which means that for J power m to go to 0 as m goes to infinity it is necessary that lambda power m should go to 0 or mode lambda is less than 1. And conversely if when lambda is less than 1 what we need to show is that this quantity will go to 0. So, that means that m choose I will write it as m choose m minus j lambda power m minus j. So, I am just replacing i with m and s, m minus j and so, this we want to show that this goes to 0 as m goes to infinity and for j equal to 1 through k minus 1 0, 1 up to k minus 1.

Now, this quantity here m choose m minus j lambda power m minus j is equal to the magnitude of so, this combination term is m into m minus 1 all the way up to m minus j plus 1 divided by j factorial times lambda power j then I have a lambda power m here. And this can be upper bounded by magnitude of m power j I am replacing all these terms with m lambda power m divided by j factorial lambda power j.

And so, basically then it suffices that to show that m power j times lambda power m. So, this is the only part that depends on the numerator is the only thing that depends on m. So, it suffices to show that lambda power m say magnitude times m power j goes to 0 as m goes to infinity. Now, there are many ways to do it, and what you can show is see this is lambda power m and this is m power j, j is some fixed number here it takes values 0, 1 up to k minus 1.

So, none of that is scaling with m. And so, this is some polynomial term this is an exponential term. And so, eventually if mode lambda is less than 1, this will eventually (())(08:48) this term and you will get 0. So, one way to do to see that is if you take logs, you have j log m plus m log lambda which we will go to so, this is m times log lambda whereas, this is j times log m and so, this will go to minus infinity if as m goes to infinity because log of mode lambda if mode lambda is less than 1 mode log lambda is less than 0.

So, basically what this shows is that even for non diagonalizable matrices, the matrix is convergent if and only if all the eigenvalues of the matrix are less than 1. So, to show this result, note that we made essential use of the Jordan canonical form.