First Course on Partial Differential Equations - II

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Lecture - 32
Wave Equation 3

Hello everyone welcome back in the previous class we derived the formula for the solution of the wave equation in 3 dimensions. So, let us again look at the formula and try to understand, some qualitative properties of the solution looking at that formula.

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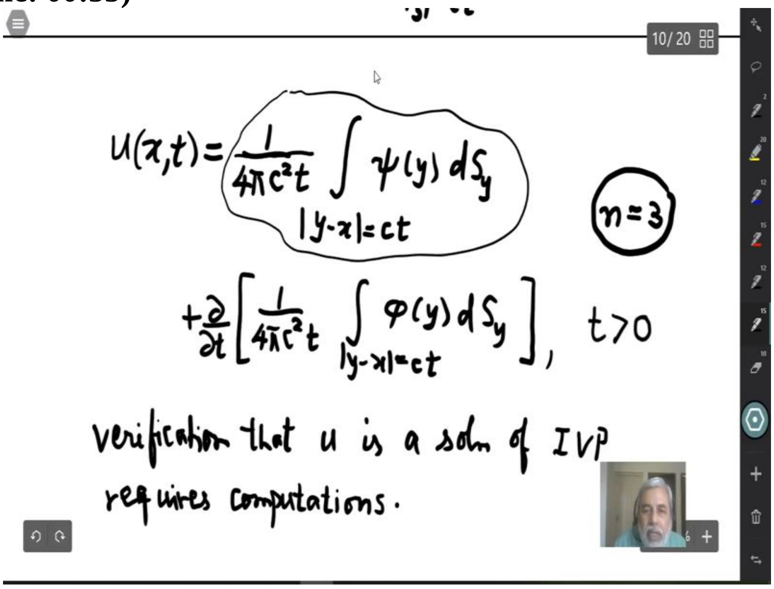

So, this was the formula we derived for the solution where phi and psi are the initial conditions u of x 0 is phi of x and u sub t x 0 is psi of x. So, compared to the n = 1 case so where it was that that the D'Alembert's formula so it was very easy to verify that u given by the D'Alembert's formula is indeed solution of the wave equation and in case n = 3 this is called Kirchoff's formula it is not very straightforward to see that u is solution of the wave equation.

So, this is the formula for t positive so it requires some computations that not difficult but one has to do that so for example it is of sufficient to verify this first part. So, this first part is a solution of the wave equation so that is sufficient for the following region.

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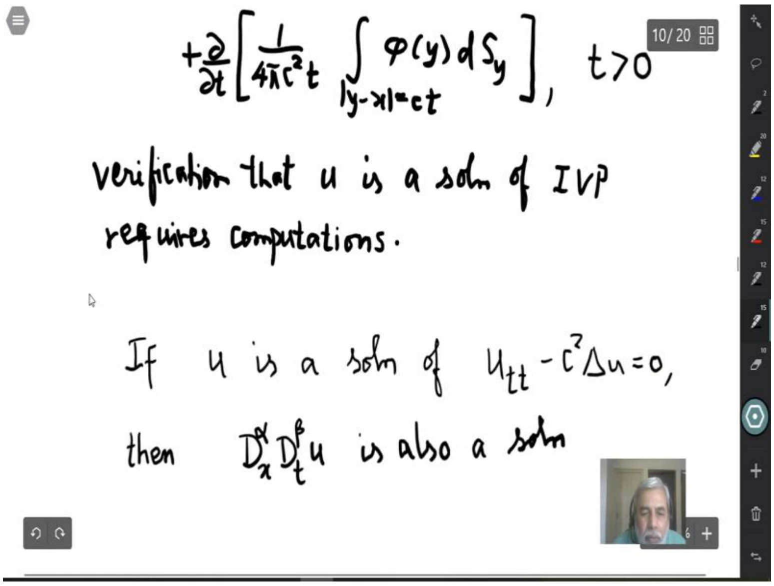

So, if u is a solution of the wave equation then if u is sufficiently smooth so we can take any number of derivatives with respect to the x any number of derivatives provided u is smooth enough is also a solution so it is sufficient to verify this one because this is just d / dt of a similar term so that proves that use a solution so that first one requires some computations I am not going to do that.

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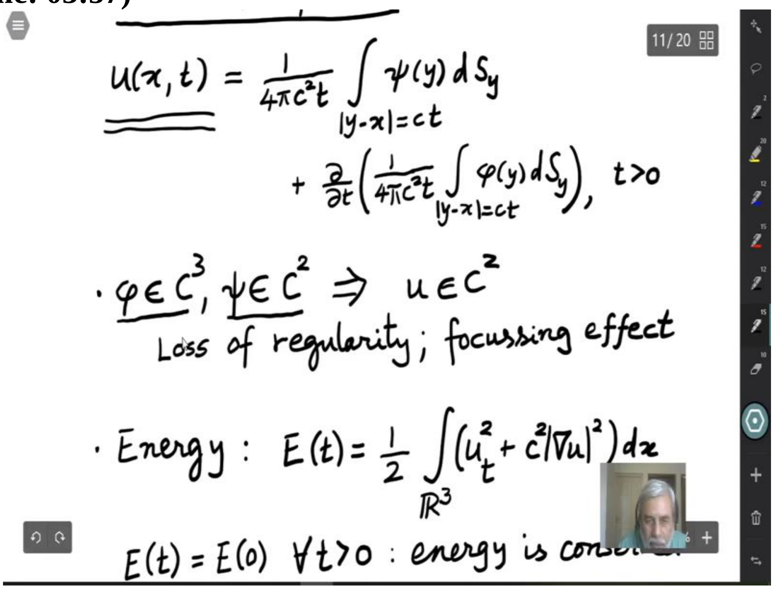

So, let us now look at some the qualitative properties coming from this solution. As I commented just now this formula is called Kirchoff's formula. So, for classical solutions we require this solution to be C 2. And this formula tells us when that u is C 2 so this is the first observation. So, if you can state this as a theorem so if the initial condition psi belongs to C 3 and psi belongs to C 2 of R 3.

So, we are just in case n = 3 so let me just repeat that because for n bigger than 3 the formula differs. So, remember again u x, 0 is psi of x and u t x, 0 in psi of x is an essential conditions so if phi is C 3 function and psi is C 2 function in R 3 then it is guaranteed that this function u defined by this Kirchoff's formula is a C 2 function. And then one verify that it is indeed a solution of the wave equation so we see that so we require more smoothness so required for psi to be C 3 for example in order to obtain a C 2 solution.

So, this is referred to a loss of regularity this we have already seen in some cases earlier also in this case it is also referred to as focusing effect and this loss of regularity in general will create problems when we want to apply this linear theory to prove the existence of solutions to the non linearity equation. On the other end so that is the qualitative behaviour as far as smooth solutions are concerned.

If you consider this energy at time t so this by definition E of t before that so in if you look at n = 1 case compare n = 1 so it is important n = 1 there is no loss of regularity but this loss of regularity persists when you move to higher dimensions. So, in comparison; if you look at this energy which is defined by half this integral over R 3 now we are just dealing in R 3. So, this same thing is also true for R n so u t square + c square grad u square mod grad u square dx. So, these are functions of x and t so we are integrating with respect to x so that is a function of t. So that is the total energy at time t provided this integrals are finite.

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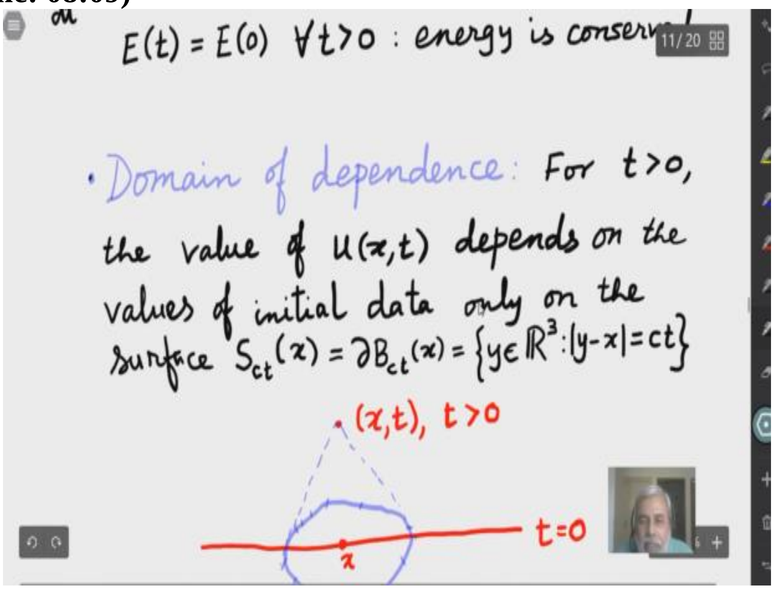

So, then a simple computation show that so this we did for it for example for n = 1 case so you can just u the wave equation and show that dE / dt = 0 the same computation. So, whatever we did for n = 1 so you simply take the differentiation under the integral sign and then do some integration parts and u satisfy the wave equation and conclude that dE / dt is 0 that means C is a function of t is a constant.

And so that is to say that the total energy is conserved so it is the same for all t. So, whatever energy we start with so that t 0 comes from the initial conditions and that will remain finite and this is a good norm to work with. So, in fact many you want linear problems with variable coefficients and even some nonlinear equations. When one is trying to prove existence of solutions such norms will be used.

And similar to the n = 1 case now we discuss again domain of dependence range of influence. So, for t bigger than equal to 0 again look at the Kirchoff's formula so take any t positive. So, the value of the solution u at x, t depends on those values of y which are lie on this field if they lie outside this sphere they do not contribute to this surface integrals. So, they do not count so that is the domain of dependence.

So, the value of the solution u at x, t it depends on the values of the initial data only on the surface the S ct. So, this notations we already introduced even in the first part so just recall that so this is this sphere centered at the x and radius ct which also denoted by the boundary of this ball this ball centered at x and radius ct and that is the explicit definition of the sphere. So this is sphere so center x and radius ct. So, you can see draw some pictures so this is t = 0 plane that is just the space not plane.

So, these R 3 and if you take any t positive so the value here depends only on this so this is S ct x the one in the blue color so that is thing the only the sphere not the interior that is important later on we see in n = 2 this is not the case. So, it depends on the entire ball so this is the domain of dependence of this point x, t; t positive.

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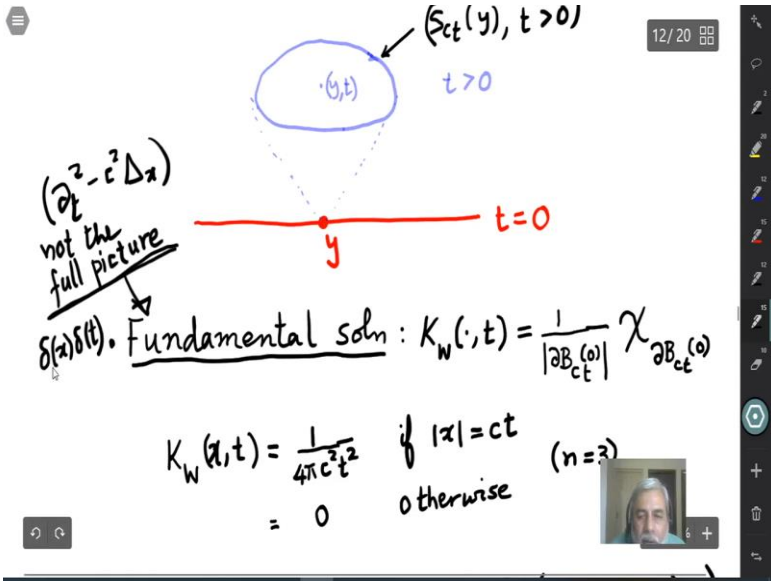

And the range of influence so; it is the reverse phenomena. So, a point in the initial space t = 0 where all that can influence the values of u of x, t for a which x it can contribute something to the value of u x, t so that is the range of influence and again looking at the Kirchoff's formula. So, this depends only on this sphere again so this y should lie on this sphere at any x. So, if y does not lie on the sphere then it will not contribute to the value of u x, t.

So, this range of influence is again a sphere t = 0 so this is the t space and this is sphere. So, this is set of all S there is t also so this is not just center is fine but there is a t positive so that is the range of S. So, now like just similar to n = 1 case so now we also discuss the fundamental solution a word of caution here in this n = 3 case and n = 1 case and in general for any n.

So, this is not a full picture of not equal picture so that you should bear in mind for the region. So, here we are concentrating only on the x variable and t we are not considering in that detail in order to apply a (())(16:02) to get the fundamental solution we have to work with not just with respect to x variable but also with t variable so, the operator is as you recall.

So, this is the wave operator so there is both the x and t variable so we have to consider this Dirac delta function with respect to both x and t and that makes computations more difficult and since we are not developed any topics in that. So, will not going to that so the main aim here is to represent the Kirchoff's formula again and n = 1 case again the D'Alembert's formula as a convolution.

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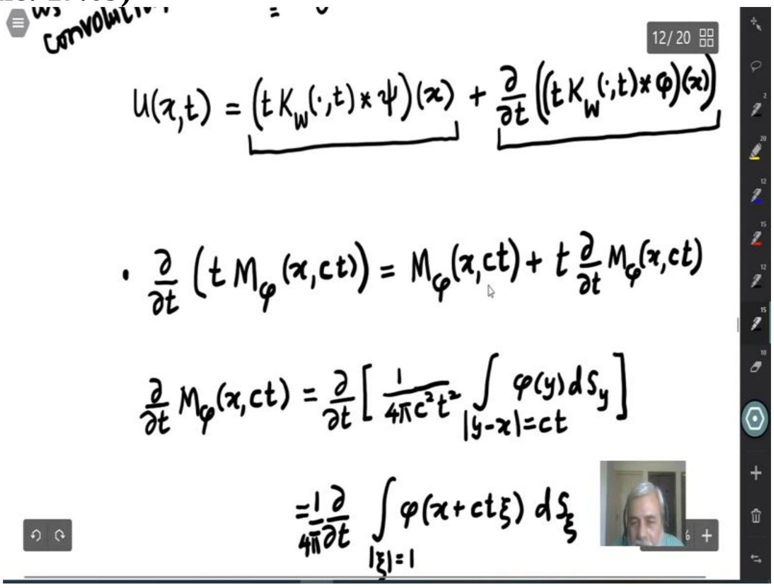

So, the main aim is so representation of the solution as a convolution very similar to what we have done for the Poisson equation and for the heat equation. So, only for that purpose we are introducing this simple form of the fundamental solution. So, this is not the full fundamental solution so that requires little more work so introduced this function K w again dot t so in case n = 1 it was characteristic function of n interval depending on t.

And in this case it is again the characteristic function but not an interval but it is a characteristic function of the sphere at the origin center at origin and radius ct and you are dividing that by the surface area. So, this is surface area again n = 3 so we come to the general case later on. So, explicit later so this K w x, t is 1 by 4pi c square t square so n = 3 that is what the surface area is and this is so if x belongs to the sphere then it is 1.

So, mod x = ct so that is the sphere 0 otherwise so it is a characteristic function and then it is easy to verify looking at the Kirchoff's formula the solution can be written indeed as a convolution so this. So, this part comes from the data psi and this part comes from the data phi. So that is the just mean aim of introducing testing so we can write the solution as a convolution so that is that thing.

So, this if you look at Kirchoff's formula again so there is a derivative term so that you can now expand and write it that is also useful. So, consider that term which is the first derivative of this tM phi M is the spherical wave function and that is simply M phi x, ct and d / dt of M phi x, ct.

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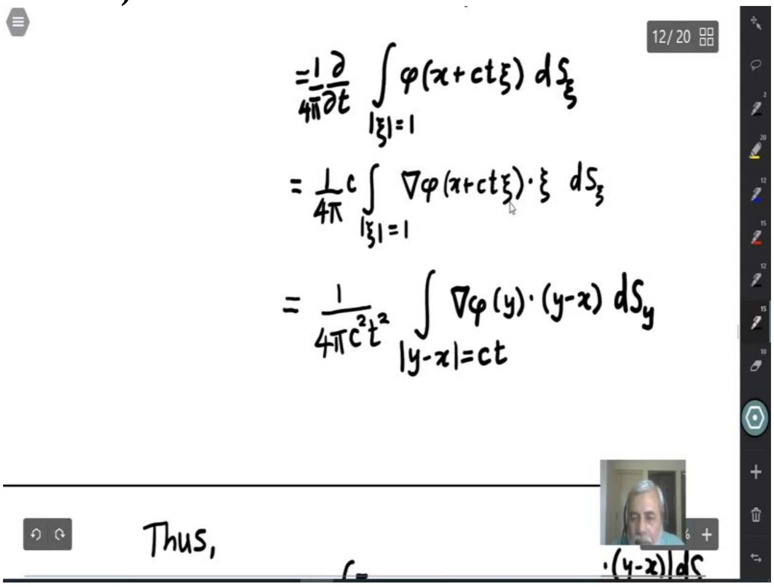

And again you use the definition of the spherical mean function. So, again there is a t here there t here so again change of variable makes it t only in the integrand and that is easy to differentiate so we can take this d / dt inside the integral and when you do that you get this nice formula and again you change the variable x + ct psi to y so you get this.

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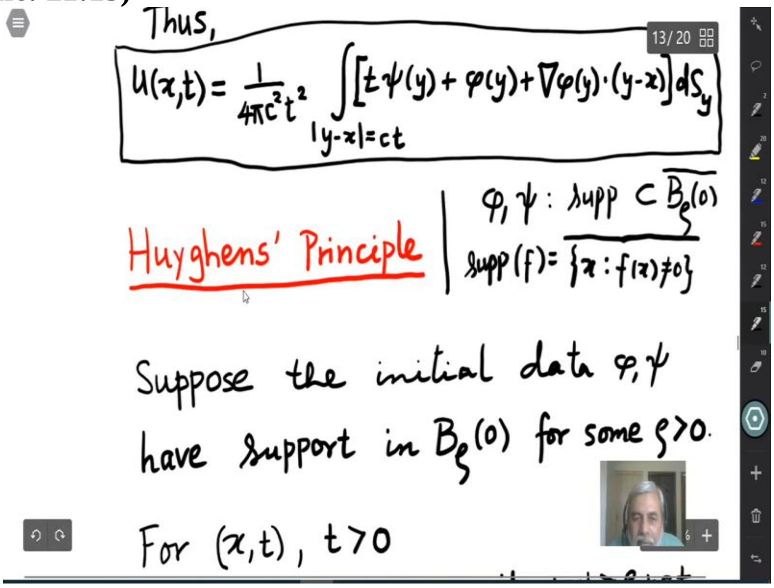

So, the Kirchoff's formula can also be written like this so without derivative so there is no derivative involved and from here also you will see that so since there is already first derivative of phi appears in the formula. So, if you want u to be C 2 we want phi to be C 3 though these 2 terms just require C 2 but this one requires phi to be C 3. So, same thing so this is now there is no derivative at all.

And now we discuss an interesting qualitative property of this solution in 3 dimensions and this is known as Huyghens' principle. So, it consults with so suppose you take phi psi such that their support is contained in B rho so support just recall support a function set of all x whichever space you are f you defined. So, f x is not 0 and you take the close set so support is always close set.

So, such functions are called functions with compact support so because this is a compact set and we would like to study what happens to the support of the solution at a later time t positive so that is the main idea behind this principle.

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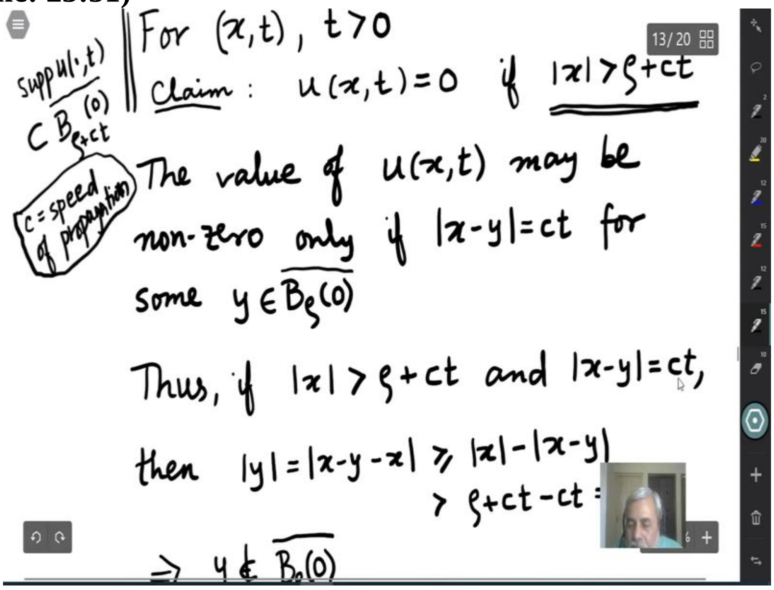

So, start with this phi and psi smooth functions so for example C 3 and C 2 which have also support in this B 0 let me put that that does not matter. So, the first claim is so you pick any point x, t x in R 3 and t positive. The first claim is this u, u the solution of the wave equation with this dot phi and psi which have supported this closer of this part so u x, t is 0 if mod x that norm x bigger than rho + ct.

So that is the first claim so that immediately implies 1 part support of u as a function of x so t is fixed is certainly contained in this rho B ct. So, if the initial data is supported in a ball of radius rho so at later time t the support of the solution is contained in a slightly bigger ball that rho + ct, rho is replaced by rho + ct and this is the reason so this c is called speed of propagation.

And this claim follows from understanding the range of influence So, the value of u of x, t this we have already seen maybe non 0 only if norm x - y ct for some y in the rho 0 anywhere but it is 0 that is this follows from the range of influence and phi and psi are only non 0 in this their support is here. So, at most they are non 0 only here in the B rho, G rho closer of that. So, if this condition holds if norm of x is bigger than rho + ct so we should show that any y sitting on this sphere centered at x lies outside this closer of the spot.

So, if we are able to show that then that will only contribute 0 value to this solution and that claim is proved. So, if norm x is bigger than rho + ct and y sits on this sphere norm x - y = ct then.

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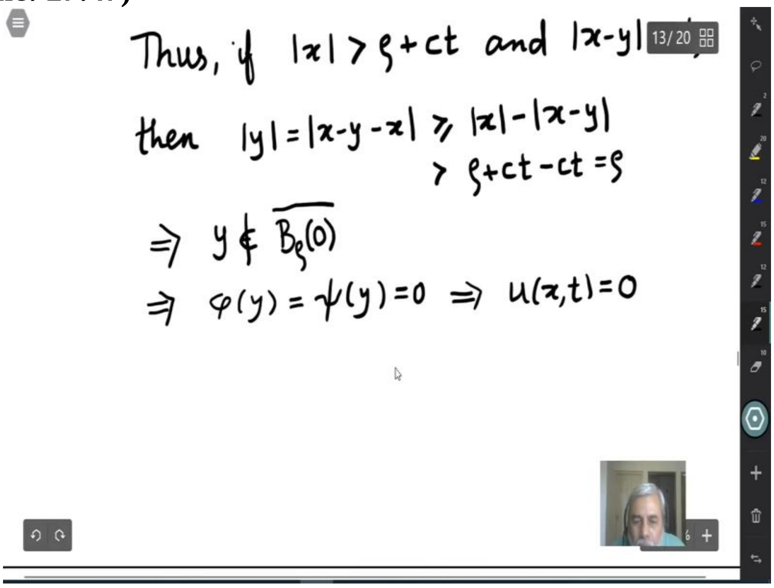

So, this is simple triangle inequality so norm of y is equal to norm of x minus so I rewrite that x - y - x and that is bigger than or equal to norm x - norm x - y. So, this is just a triangle inequality and now we are assuming norm x is bigger than rho + ct and y sits on this sphere so that is minus ct so this norm y always bigger than rho so phi and psi 0 there so you u x, t is also 0. So that is with the claim but more is true that is what we will saw the support is still smaller than this.

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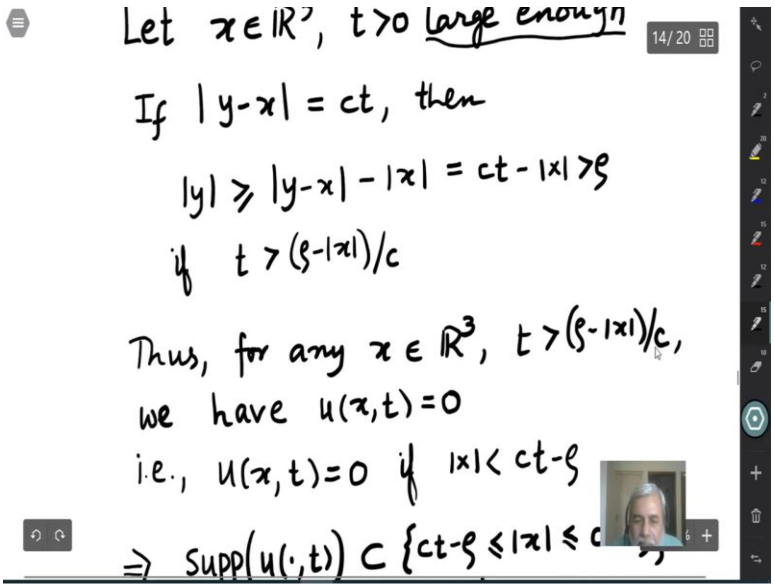

So that is what we want to show now so pick again any x in R 3 and t positive large enough so what is that large enough we will see in a moment. Again you consider any y on this sphere because that is the point where the initial data influences the value of u of x, t. So, if norm y - x is ct that is y sits on the sphere. So, again by triangle inequality so norm y is bigger than equal to norm y - x - x now just interchange the sign here.

And this norm of y - x we are assuming ct and this x as it is and if t is large enough we can make it bigger than rho so that is why it is large enough. So, this so if t is bigger than this then you also know that this norm y bigger then rho again it will contribute only 0 value to the solution and that is what we write here. So thus for any x in R 3 and t bigger than rho - norm x by c then we have u x, t 0.

So, first to prove that for any t positive u x, t 0 if norm x is bigger than rho + ct and now we have proved if t large enough u x, t is also 0 if norm x is less than ct - rho. So, this so far t large enough the support actually lies in this annulus.

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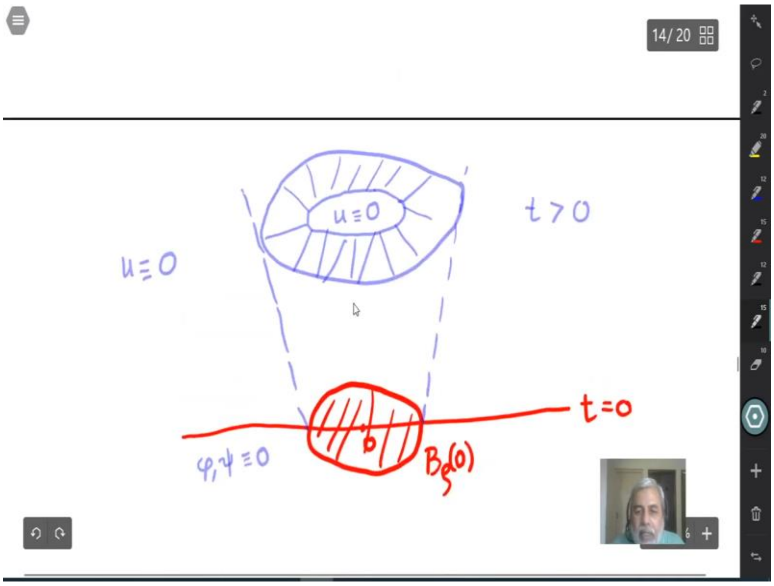

So, this is what I have written here so this is the support of initial data closer of is B rho 0. And at a later time t, t large enough we have the support of u is only in this shaded region. So, u 0 inside here and u 0 outside there so the support is in the annulus.

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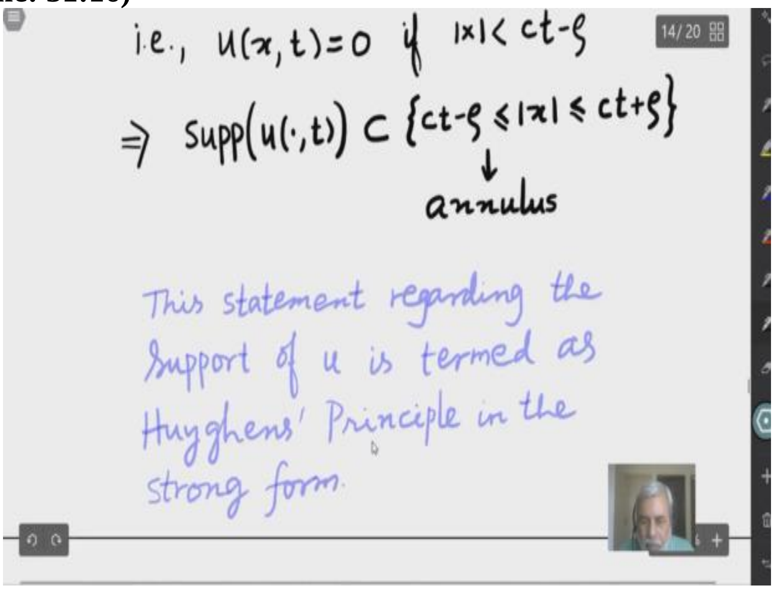

And this statement regarding the support is referred to as Huyghens' principle. And physically it is interesting to observe here so the sound propagation is very well modeled by this wave equation. And we are in dimension 3 we are living in 3 dimensional space for the reason. Whatever I am saying you will hear it after some time and then you will not hear it again after some time.

So that it only leaves for a short period of time so otherwise we will all be listening to whatever we have heard since long. So, thankfully so this sound propagation is very well modeled by this 3 dimensional wave equation. So this is in very much contrast to it n = 2 case where you do not have this Huyghens' principle in the strong form. So, this is the strong statement you have only Huyghens' principle in the weak form.

So that is the difference between n = 3 and n = 2 and that persists in all higher dimensions n odd dimension and n even dimensions though physically you do not have any example. But mathematically you will see that statement so this support statement is different from odd n and even n that we will see.

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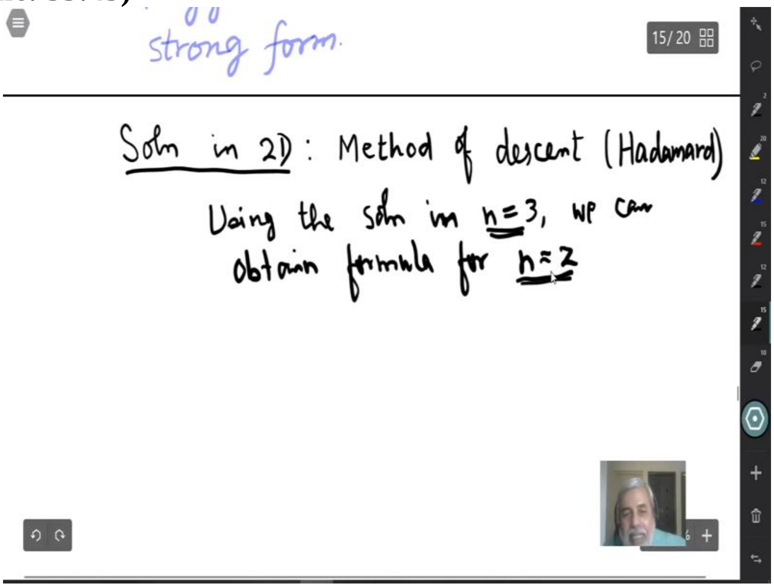

So, our next goal is to obtain the solution for the solution in 2D and surprisingly we do not have to do much work and so this is referred to as a method of descent and this is due to Hadamard. So, it is impact called Hadamard methods of descent. So, using the solution in n = 3 so that we have already derived in the Kirchoff's formula we can obtain formula for n = z so, for higher dimension so we can descent to a lower dimension.

So, unfortunately we cannot do this there is no method of ascent. So, from n = 1 we cannot build though we huge the case of n = 1 to derive solution for n = 3 but after much work but here you see that there is absolutely no work just you use the formula for n = 3 and take some special initial data to derive the formula for the n = 2 and that is what we will discuss in our next class. Thank you.