**Partial Differential Equations - 1**
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**Lecture 41**
**One Dimensional Wave Equation**

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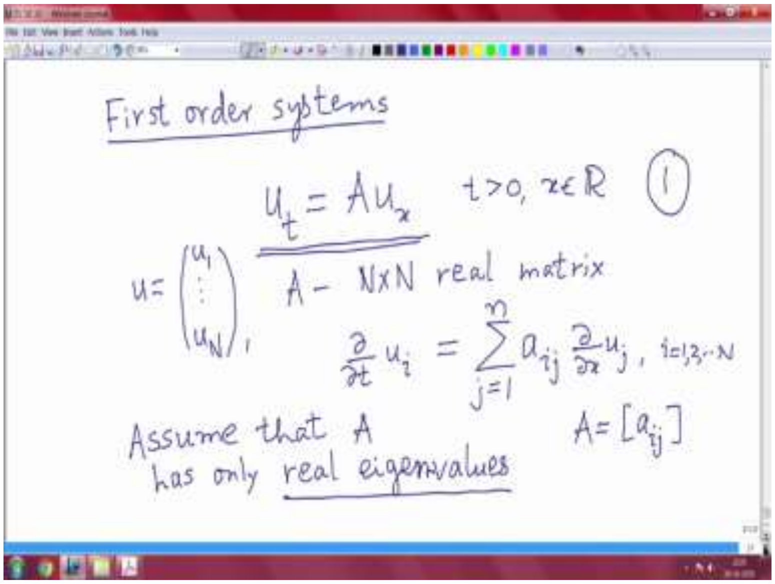


Welcome back. Today, we will continue discussion on some related topics of One D Wave Equation. So briefly discussed first order systems. So now we consider such a system, let me write equal to, so again t is a positive real number and x varies over the real line, so instead of first order system in one space variable.

So here u is a column vector, un, and A is N by N real matrix. So if you expand this, so this we have written in matrix notation, if you write component wise, then it is d by dt of ui is equal to summation j equal to 1 to n aij, d by dx of uj. So there are N first order equations, so that is why it is first order system. So aij are the components of the matrix.

We assume that, so this is basic assumption, assume that A has, the matrix A has only real eigenvalues. As you all know even a real matrix can have complex eigenvalues, so this is an assumption we are doing on the matrix, given matrix here. With these assumptions, so let me call this system as 1.

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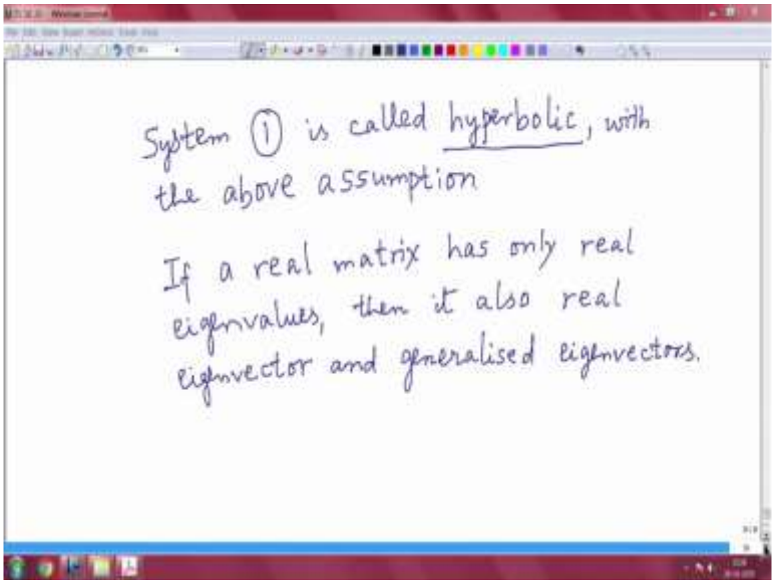


So system 1 is called hyperbolic, with these assumptions, so that is important with the above assumption. Further classification of system 1 is possible depending on the nature of eigenvalues and the corresponding eigenvectors, that really takes us into some deeper aspects of the theories which I am not going to do. For the purpose just illustrating the applicability of several methods we did in this chapter, I just restrict myself to the case when A has just eigenvalues.

I will not even assume that they are distinct or whatever, just real eigenvalues. In that case again, from linear algebra we learn that, so if a real matrix has, so this is from linear algebra I am recalling, has only real eigenvalues, then it also has real eigenvectors and generalized eigenvectors. So it is good time to recall all these notions from linear algebra, generalized eigenvectors. So that is from linear algebra.

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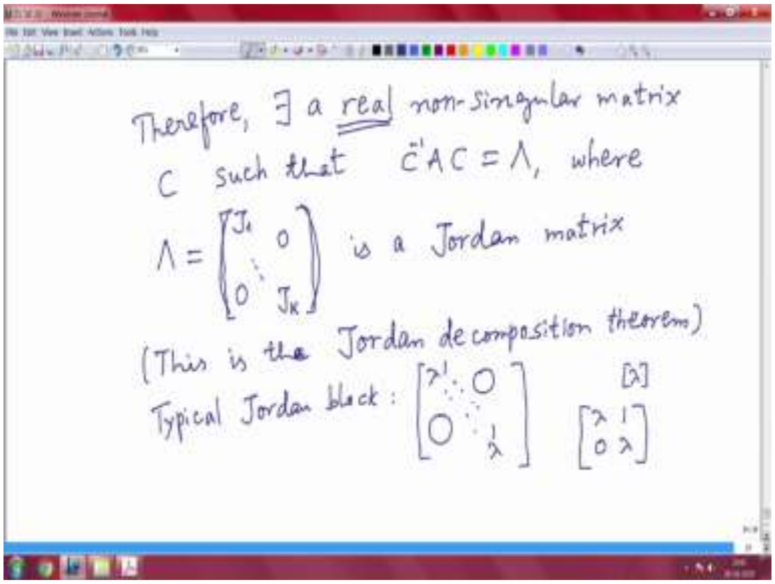


So therefore, there exists a real non-singular matrix, C such that C inverse AC is Jordan matrix, call it Lambda, where Lambda, we usually write it as J1, Jk is a Jordan matrix. So this is the Jordan decomposition theorem. So this is one of the important and very deep theorem of linear algebra. So it is important that with the assumption that A has only real eigenvalues, we are going to get a real non-singular matrix, that is important for us, because we are looking only for the real solutions.

So this J1, J2, Jk are Jordan blocks, a typical Jordan block, let me write, typical Jordan block is a square matrix, I am using both, so this is also a matrix notation, so let me just write, square brackets. So on the diagonal you have same Lambda and all the elements below the diagonal are 0 and just one on the diagonal, just above the diagonal, sub-diagonal and then again 0, very simple form. It could be 1 by 1, so in that case you have one, just Lambda.

So for 2 by 2, so we get Lambda 1, 0m. So these are typical examples, so this is 1 by 1 Jordan block, which is just one element and this is 2 by 2. So what is the advantage of this? The advantage of this is the system 1 reduces to a simpler form which can be easily analyzed.

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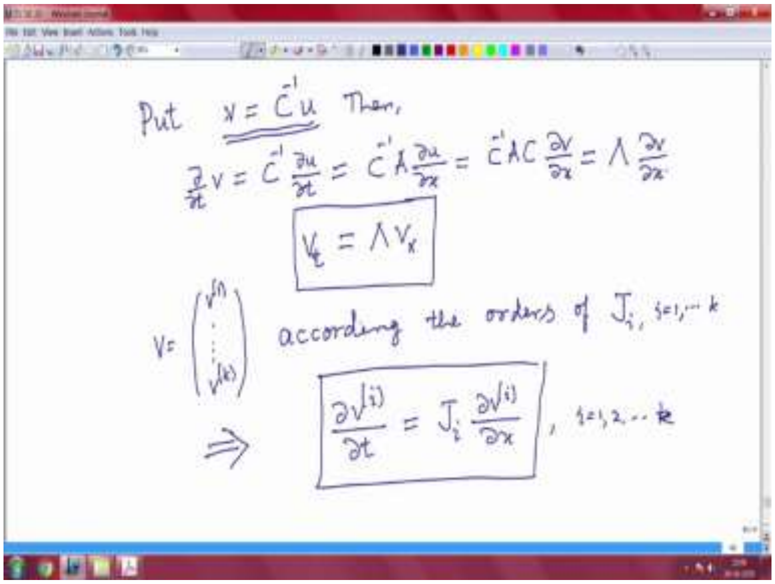


So put v is equal to C inverse u. Then let us compute d by dt of v, so since C is constant matrix, so we can just bring it out and use system 1, so this is C inverse, A du by dx. And again, you write in terms of v, so C inverse is C, so us is Cv, so I write it here dv by dx. And this by Jordan decomposition theorem that is expressed as a Jordan matrix, so del v by del x.

So this change of variable reduces the system 1 to a simpler form. So namely, so let me again write it, so vt is vx. So now if you look at the structure of Lambda, so they are all Jordan blocks, so we decompose v as v1 et cetera vk according to the orders of the matrixes Ji, i equal to 1. So for example J1 is 2 by 2, then this v1 will have two components.

So since this implies, we can write down a system for each of these components, so del vi by del t is Ji del vi by del x, so i is equal to 1, 2, k. So in this way, we have partially decoupled the system 1 into more or less diagonal, so if for example, A is diagonizable, then this Lambda would be a diagonal matrix and then this will be a completely decoupled system.

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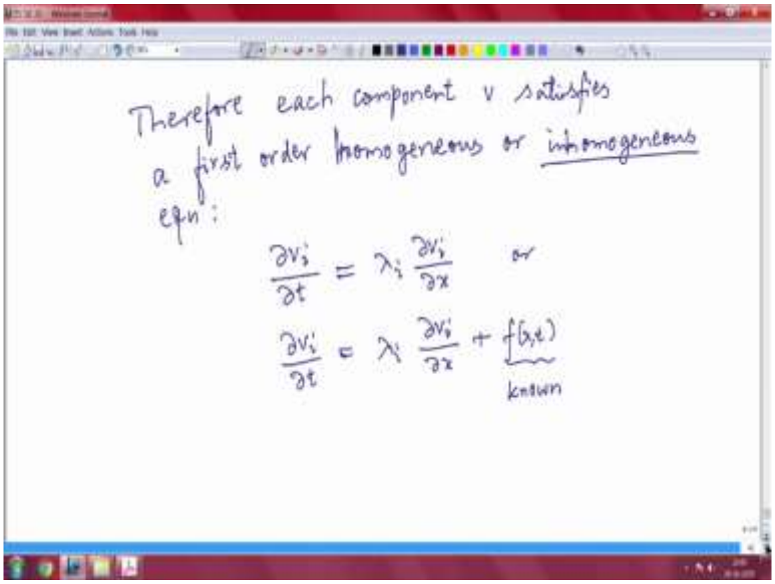


So if we again write, again look at the typical Jordan block, it is of this form, so each component, therefore each component of v satisfies a first order homogeneous or inhomogeneous equation. So let me just write one component vi, so this del vi by del t equal to Lambda I, del vi by del x or del vi by del t equal to Lambda i, plus something, no, inhomogeneous. This is a function of xt, no. So in either case we can just use the method of characteristics and write down the solutions.

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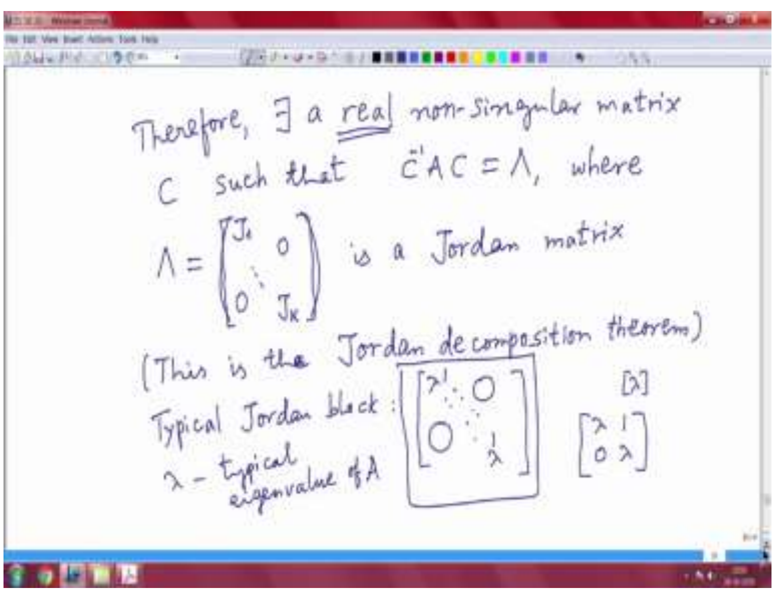

So again, remember, so I did not stress that, so this Lambda's, this Lambda, so this J1, J2, Jk, so each one will look like this, there is Lambda there, so they involve different eigenvalues of Lambda. So Lambda is a typical eigenvalue of A. So they all come there, they all come in the diagonal of this matrix Lambda, so they are all eigenvalues of A.

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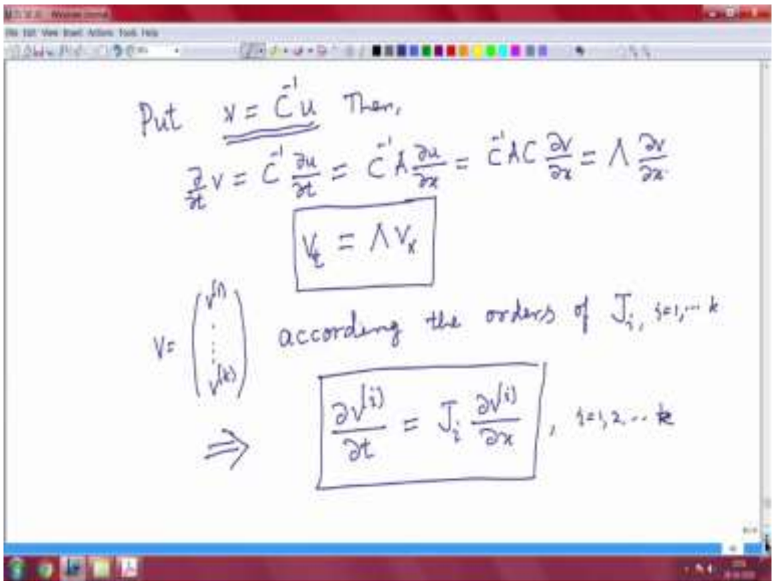


So in this context, the eigenvalues in this context the eigenvalues of A play the role of speeds of propagation. Let me now present some examples of various things we did in this one D wave equation.

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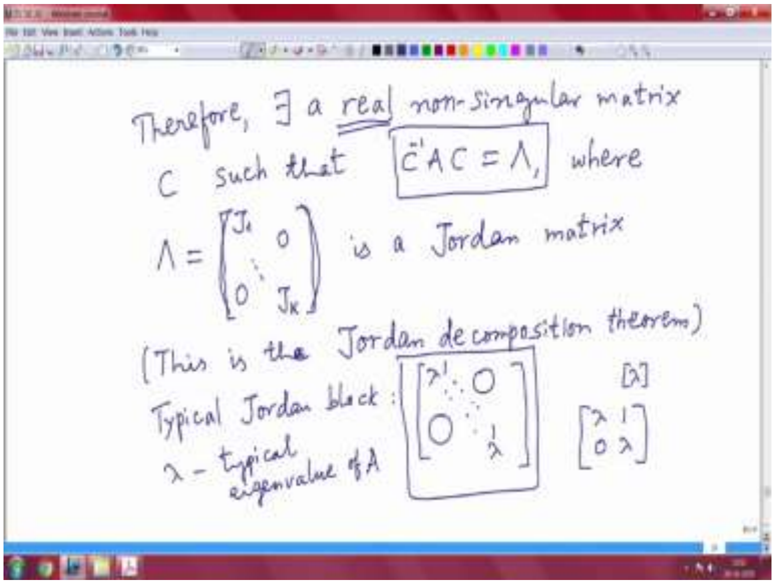


So in general, the computation of the matrix C and obtaining this Jordan form is a very difficult task, so it is more or less easily done for 2 by 2 systems, even 3 by 3 systems okay, but once you go beyond third order, then computations become very heavy, but in principle, as far as theory is concerned, it is very nice lifts here and we can decompose the given first order system into more or less a diagonal form.

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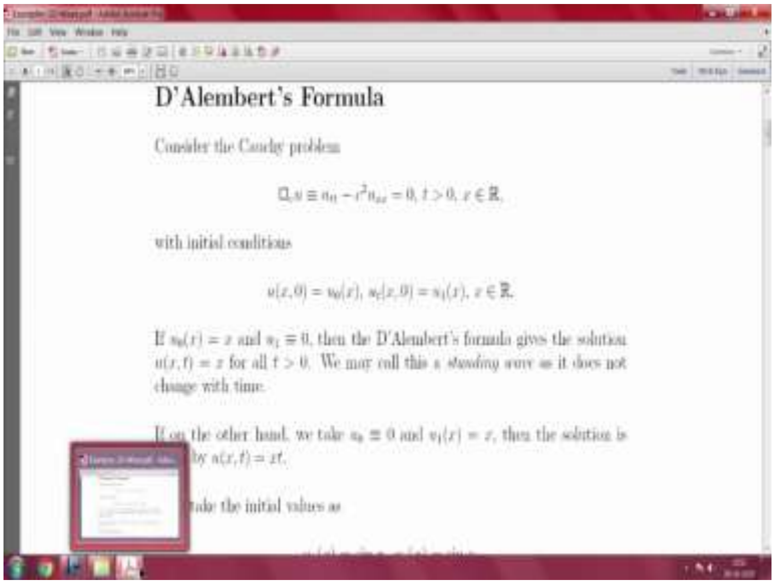

Consider the Cauchy problem

$$\Box_c u \equiv u_{tt} - c^2 u_{xx} = 0, \; t>0, \; x\in\mathbb{R},$$

with initial conditions

$$u(x,0)=u_0(x),\; u_t(x,0)=u_1(x),\; x\in\mathbb{R}.$$

If $u_0(x)=x$ and $u_1\equiv 0$, then the D'Alembert's formula gives the solution $u(x,t)=x$ for all $t>0$. We may call this a *standing wave* as it does not change with time.

If on the other hand, we take $u_0\equiv 0$ and $u_1(x)=x$, then the solution is given by $u(x,t)=xt$.

take the initial values as

$$u_0(x)=\sin x,\; u_1(x)=\sin x.$$

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If we take the initial values as

$$u_0(x)=\sin x,\; u_1(x)=\sin x.$$

then the D'Alembert's formula gives the solution as

$$u(x,t)=\frac{1}{2}(\sin(x+ct)+\sin(x-ct))+\frac{1}{2c}\int_{x-ct}^{x+ct}\sin\eta\,d\eta$$

$$\Box_c u \equiv u_{tt} - c^2 u_{xx} = 0,\ t > 0,\ x \in \mathbb{R}.$$

with initial conditions

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \mathbb{R}.$$

If $u_0(x) = x$ and $u_1 \equiv 0$, then the D'Alembert's formula gives the solution $u(x,t) = x$ for all $t > 0$. We may call this a *standing wave* as it does not change with time.

If on the other hand, we take $u_0 \equiv 0$ and $u_1(x) = x$, then the solution is given by $u(x,t) = xt$.

If we take the initial values as

$$u_0(x) = \sin x,\ u_1(x) = \sin x,$$

then the D'Alembert's formula gives the solution as

So let me show you some examples. So various I have, so let me start with D'Alembert's formula, so this is wave equation we studied this far with given initial data. So here I have taken very simple initial values, for example, if you take u0 equal to x and u1 identically 0 and then the D'Alembert's formula give the solution u xt equal x for all t positive.

So we call this standing wave as it does not change with time, so important here, so in this case it just remains x. So if I now take u0 identically 0 and u1 equal to x then the solution is given by xt, x times t. And similarly if you take these as initial values, u0 and u1 both sin x, but they are arbitrary, we can take any c2 function and c1 function respectively. So here I have just take the simple function, so that we can do the computations easily, that is the only reason.

So you should practice with lots of simple examples where you can integrate easily. So in this case, so D'Alembert's formula gives us u xt equal to sin of x plus ct plus sin of x minus ct plus 1 by 2c integral x minus ct to x plus ct sin Eta d Eta. So if you work out that then you get the solution as sin x plus sin x cos t, plus c inverse sin x sin t.

So you also, I did not mention the inhomogeneous equation here, so you should also try that, so taking instead of 0 you take some F of xt where you can integrate it easily and see how the Duhamel’s principle works and write down the solution. So this is for the D'Alembert's formula. And then we discussed equations with distinct roots and also multiple roots.

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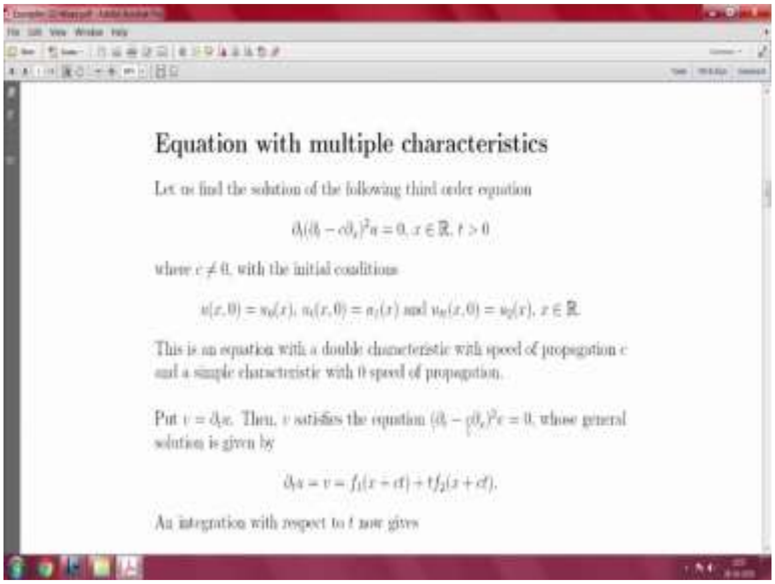


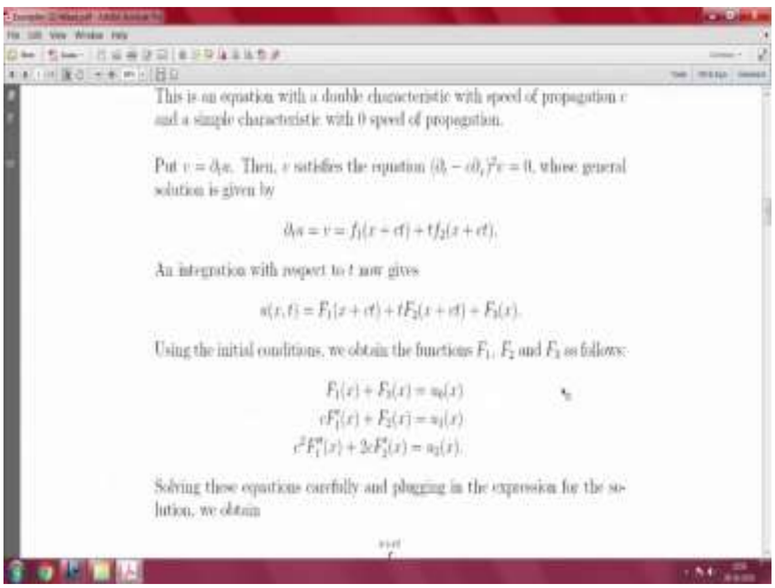


So here I have taken a simple example of third order equation. So as you see here, so this is a double characteristic part, so del t minus c del x square equal to 0 and here there is only del t. So we can say that one speed of propagation, simple speed of propagation 0 and double characteristic with speed of propagation c, taking c not equal to 0, otherwise it will be just del t cube, so just an illustration how the solution looks like in comparison with D'Alembert's formula.

So one way of obtaining the solution is substitute, make the substitution v equal del tu and the v satisfies this double characteristic equation and this we have already seen that a general solution for this second order equation is F1 x plus ct plus t F2 x plus ct. And that is del tu, so we obtained uv really is nothing but del tu. So now it is only differentiation with respect to t, so we just integrate that.

We integrate this del tu equal to this expression and now you realize that this u is given by F1 x plus ct, so you integrate only with respect to t, so x remains the same. And this F3 x is added because that is constant as far as t derivative is concerned. So instead of just a constant we can add a function of x because it is differentiation with respect to t.

And now you plug in the given initial data, so ux 0 equal to u0 x, the first derivative with respect to t at x0 is u and x and second derivative with respect to t is u tx, and if the substitute in this general formula, general expression of the solution, then we get three equations. They are simple, linear but you have to really be careful in determining this F1, F2, F3.

And once you determine F1, F2, F3 you plug in, in this general expression. So I have left several steps for you to work out and this is the final solution of that equation. So you see that because of this double characteristic nature of the equation, we require more smoothness of the initial condition in order to obtain less differentiable solution.

Again, loss of regularity, so we require u0 and u2 to be c2 functions, whereas u1 is required to be a c3 function, and then the solution has required smoothness namely twice differentiable with respect to x and thrice differentiable with respect to t, so remember here, the equation. So we require u to be differentiable with respect to t three times and with respect to x only two times.

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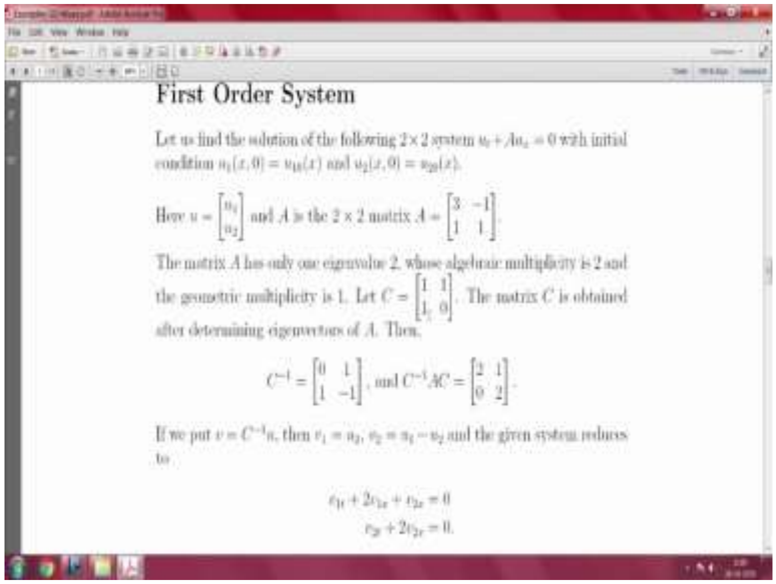


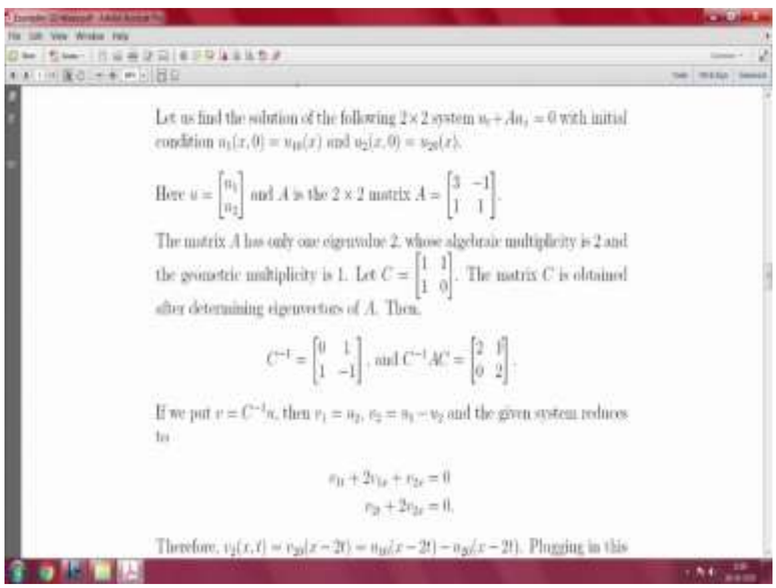

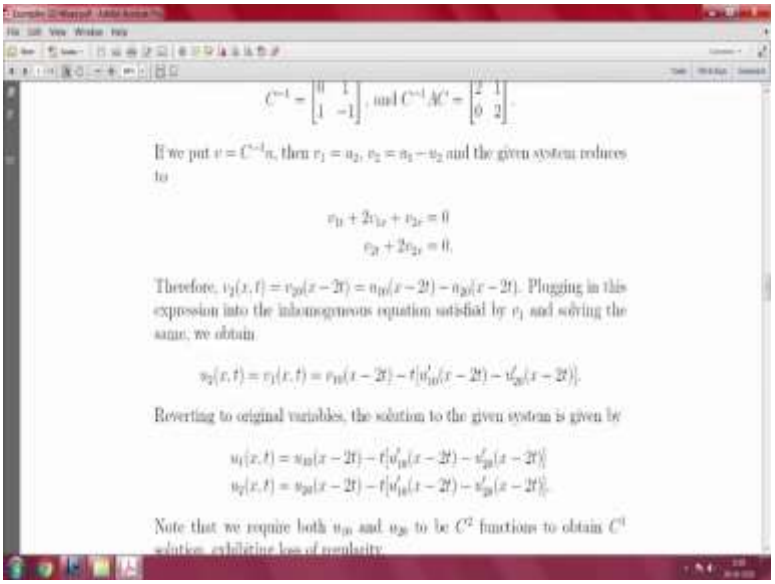

$$C^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}, \text{ and } C^{-1}AC = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

If we put $v = C^{-1}u$, then $v_1 = u_2$, $v_2 = u_1 - u_2$ and the given system reduces to

$$v_{1t} + 2v_{1x} + v_{2x} = 0$$
$$v_{2t} + 2v_{2x} = 0.$$

Therefore, $v_2(x,t) = v_{20}(x-2t) = u_{10}(x-2t) - u_{20}(x-2t)$. Plugging in this expression into the inhomogeneous equation satisfied by $v_1$ and solving the same, we obtain

$$u_2(x,t) = v_1(x,t) = v_{10}(x-2t) - t[u'_{10}(x-2t) - u'_{20}(x-2t)].$$

Reverting to original variables, the solution to the given system is given by

$$u_1(x,t) = u_{10}(x-2t) - t[u'_{10}(x-2t) - u'_{20}(x-2t)]$$
$$u_2(x,t) = u_{20}(x-2t) - t[u'_{10}(x-2t) - u'_{20}(x-2t)].$$

Note that we require both $u_{10}$ and $u_{20}$ to be $C^2$ functions to obtain $C^1$ solution, exhibiting loss of regularity.

And just now we discussed the first order system, so here I have taken a simple 2 by 2 system. So here matrix is 3 minus 1, 1, 1, A has only one real eigenvalue 2, so that is real eigenvalue, so this equation is hyperbolic, whose algebraic multiplicity is 2 and geometric multiplicity is 1 and let us see this matrix. So this computation of the C how you obtain C, you have to work out the eigenvectors and generalized eigenvectors.

And then we get C inverse AC is just this Jordan block, 2, 2, 1, 0. And now as we did so we write the v equal to C inverse u and so the two components of v1, v2 of v they satisfy these two equations. So equation for v2 is a first order equation, which is homogeneous, whereas for v1 it is inhomogeneous where v2 appears as inhomogeneous term.

But v2 determined by the second equation, so that v2 is norm. So, if you work out again by method of characteristics, so I have written here the final solution.

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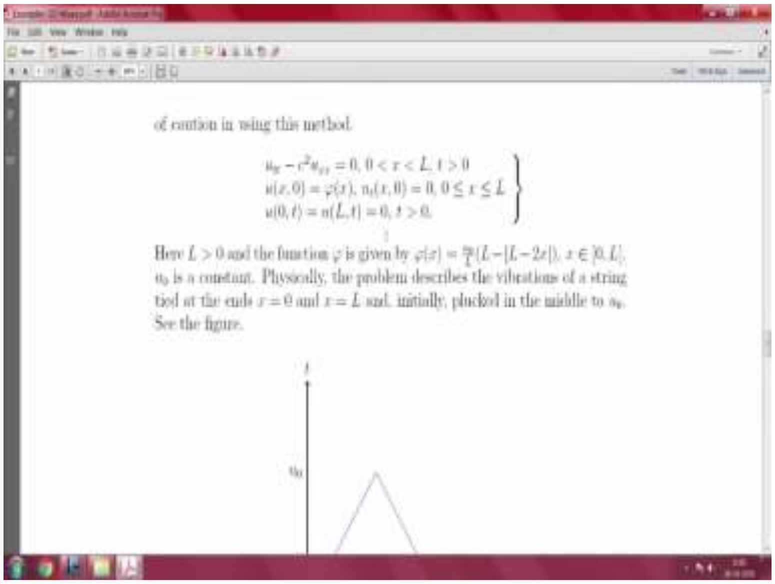


Let me just complete with one Fourier method, so this is taken again a simple example here, so wave equation in a finite interval with initial conditions on the interval 0L and then boundary conditions on x equal to 0 and x equal to L. So here I have taken this, so physically, this describes the motion of a string which is tied at the ends, 0 and L, so here it is 0 and L and plugged in the middle to a distance of u0, it can below or above, so I have just drawn here a typical method separation of variables follows.

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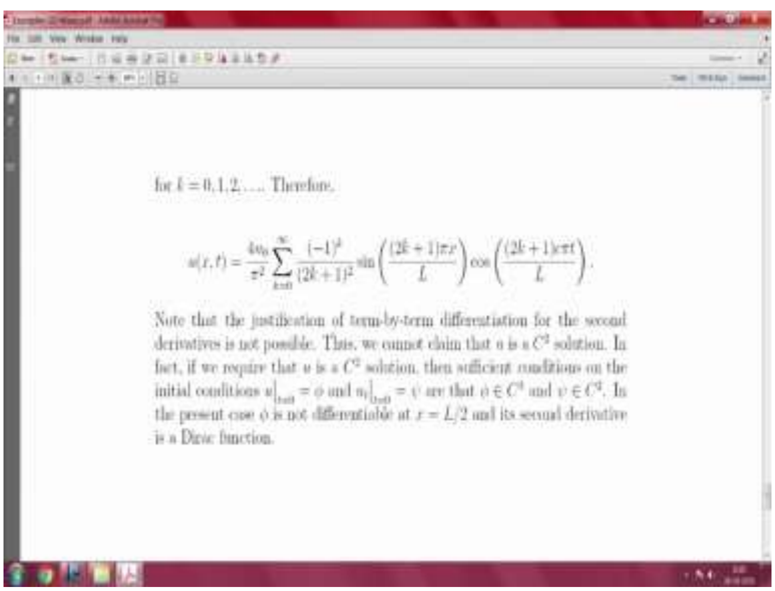

So if you work out, so just look at the work out here. so my comment will be on the final solution. So if you work out in this particular case, so we have to do some hard integrals here, they are not so hard, but still one has to work out and the final solution in this case, you get this.

So remark is that u is not a c2 solutions, so this is limitation of the separation of variable method, so though a nice looking formula we cannot claim that u is a c2 function, because generally in this infinite series we rely on term by term differentiation and if u is c1, that is no problem, because if you differentiate once, so still there is one factor left here 2k plus 1 and that makes that summation conversion.

But when you take the second derivative term by term differentiation, this 2k plus 1 square vanishes completely, so we are just left with sin and cos sin, and that we cannot make out whether convergent or not and it is not. So u is not a c2 solution. So in general, you require more smoothness on the initial conditions in order that the separation of variable methods works.

So you should keep this mind whereas the D'Alembert's formula and similar things we described yesterday, so they do not more smoothness, but of course we cannot obtain a solution in a closed form. That is the drawback there, but the drawback here is this one. So, with this I conclude this chapter on one dimensional wave equations.

So in these few lectures we have covered some aspects of one d wave equation. So in such a short span it is not possible to cover all the aspects of even one d wave equation, but since we would like to show you all the important equations, namely equations, Laplace equation and heat equation. So we will do these things in more detail maybe in a next course. Thank you.