First Course on Partial Defferential Equations – 1

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Lecture – 22

Laplace and Poisson Equations - 5

Okay welcome back. So in the previous class we have proved the mean value property that if u satisfies harmonicity, then it satisfies the average, every value can be obtained by looking at the average, either as a surface average or a volume average.

Then we had stated a theorem, the converse is also true. That means if u is a C2 function and satisfies the mean value property, then u is harmonic and I made some final remarks that the C2ness is not really needed, but the proof is different. If you assume proof that u is C2, the proof is quite easy which I am going to do now.

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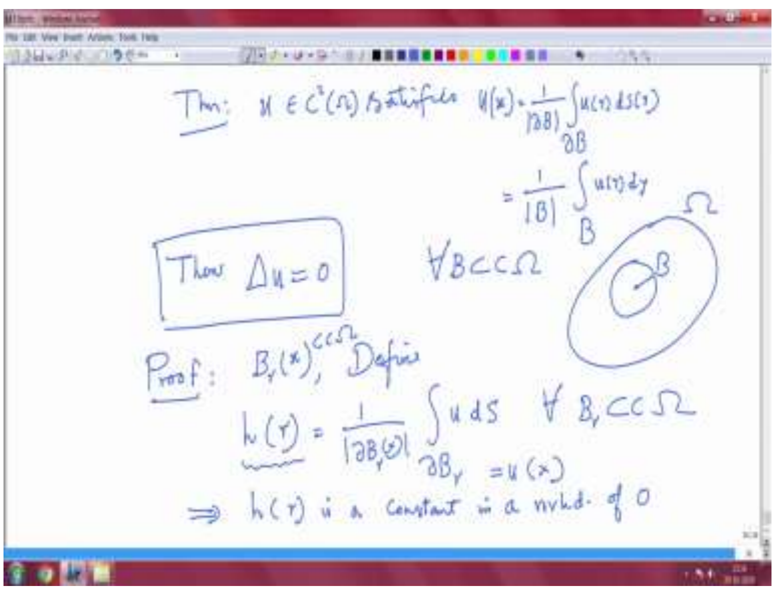

So let me recall once more the theorem stated earlier so u belongs to C2, omega satisfies u of x, u of x or u of y whatever it is does not matter, say u of x equal to 1 over boundary of b, integral of boundary of b, I interchanged the variable, does not matter. That is the same as 1 over b, integral over b, u of y, dy. And this is for all, you take any point here, you take any ball, does not matter, any ball b here. Okay whatever be the radius.

This should happen for all b compactly contained in omega. That is important. Then Laplacian of u equal to 0. That is the conclusion. So let me prove, give a proof, proof is quite good, easy, not very difficult at all. So you look at the ball of radius r centered at x that is what we will do it. Look at here and then define h r is equal to, define a function h r is equal to the ball of radius r so you have your h r, modulus of d b r of, any ball, b r of x, b r of x is compactly contained in omega, integral d b r into u, ds.

And this r h s, so this is true for all b r, right, b r contained in omega. That means this is independent of the ball. You are defining x so this is the same value that is given to be u x. That means this implies h r is constant, h r is a constant function in a neighborhood, that is all, constant in a neighborhood of 0. That is it. So constant function in r is, when r is big the ball will go far away, but we are only bothered about the neighborhood of r.

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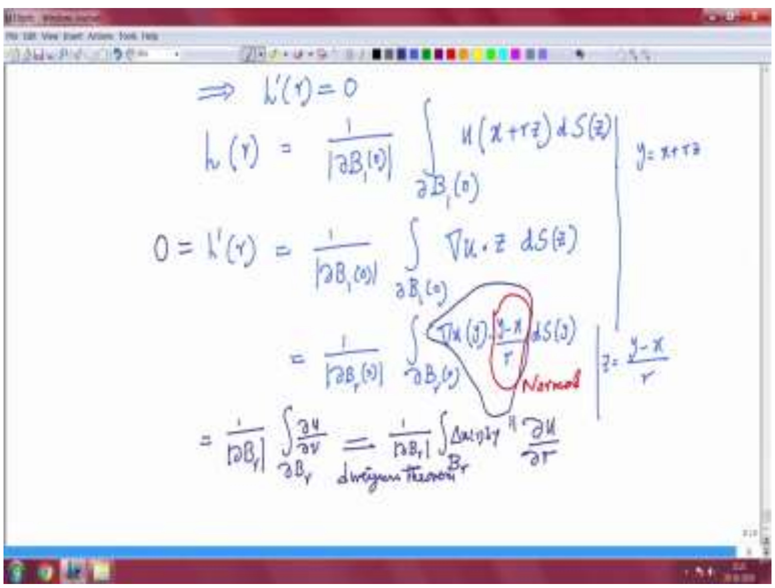

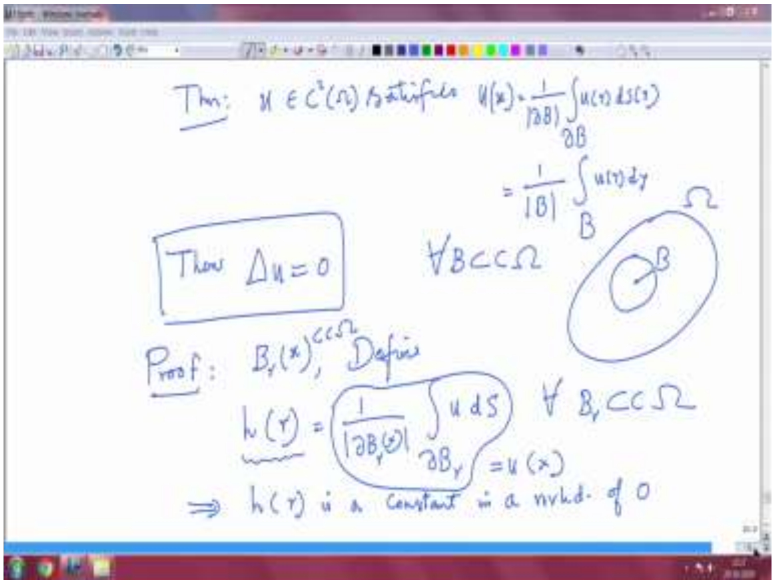

So that implies the derivative of h prime of r equal to 0. So since h is a constant function, now we compute h r in a different way. So let us try to compute h r, h prime of r. So you have h r so recall your h r once again so let me go to, let me not write. So you differentiate. When you differentiate here as I said the integral is with respect to r. There is a variable here. These are all depends on r now. So you cannot take the differentiation immediately inside.

What you do is that you again make a change of variable of translation and dilation. When you make a translation and dilation, you can write h r to be the same way. You can write, it will take care of the parameter correctly. It is the same as all this parameters will come correctly. We have done it some one or two or three occasions so we will not repeatedly do again but we will be using it again and again.

B 1 of 0, so you make this translation, u of, now x is the point so u is x here plus some Tau z. Okay so let me not get confused with u. Some variable, does not matter, so h r. So it has to be r z, d s z, any variable you can use it. So I am making a change of variable here. Okay so with change of variable this will be this one, this is only boundary.

Now I can differentiate so therefore h prime of r is equal to, now it is all independent of r, the b 1 of 0 integral of d b 1 of 0. Now it is a multi variable function. So you know how to differentiate with respect to this variable r so first differentiate with respect to all the variables x i that will produce you grade x and then differentiate with respect to r that z is a vector now because it is in the ball. So grade u dot z is equal to ds of z, so you get that.

Now come back to the original thing. If you come back again, make a dilation and translation in the other direction, this will become b r of x, again at the point you get b r of x, integral of, I am repeatedly using the same thing which you have, that thing. Now you will have grade u back to y. So your change of variable in this direction is y is equal to x plus r z.

So from here you are again going back so doing the same variable now you are, so this is x here so you have the other way you do this one, same thing. So your z will be y minus, that is your thing, y minus x, y r, ds of y. You see, x is fixed so it is a change between y and thing. So here y minus x, that is your change of variable. But what is this? This is nothing but your normal. This component, this is your normal.

Once that is normal, your this one, so this one is your normal derivative, d u by d r and normal is d u by d r so we can write this d u by d r. So you have this equal to 1 by modulus of d b r, integral of d b r, d u by d r. Now apply your divergence theorem. Once you apply your divergence theorem this will be 1 by modulus of d b r, integral of Laplacian d b u of y, dy over your b r. So what did you get now. So let us, and this is 0, so you already know this is 0.

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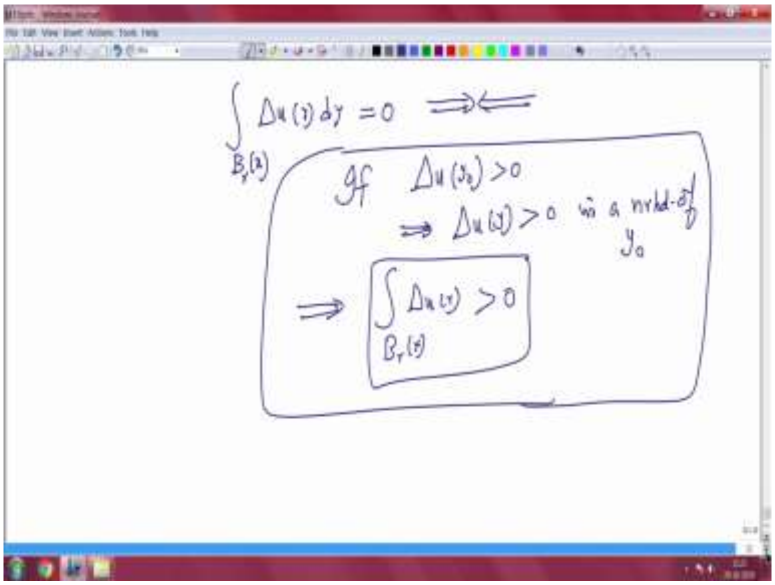

So that implies integral of Laplacian of u of y, d y for a ball of radius r around x is equal to 0. Now that gives you a contradiction because this is a contradiction. Why this is a contradiction? If Laplacian of u at y some point y naught is positive by continuity will imply Laplacian of u of y will be positive in a neighborhood of y naught.

That will imply integral of Laplacian of u of y, some neighborhood, for some r, this will become, so that is the reason it is a contradiction. So you have your result that the converse to the mean value property. So we have, and this mean value property is extremely useful in the proof of many things and you can derive many results. As we go along we may see some of them.

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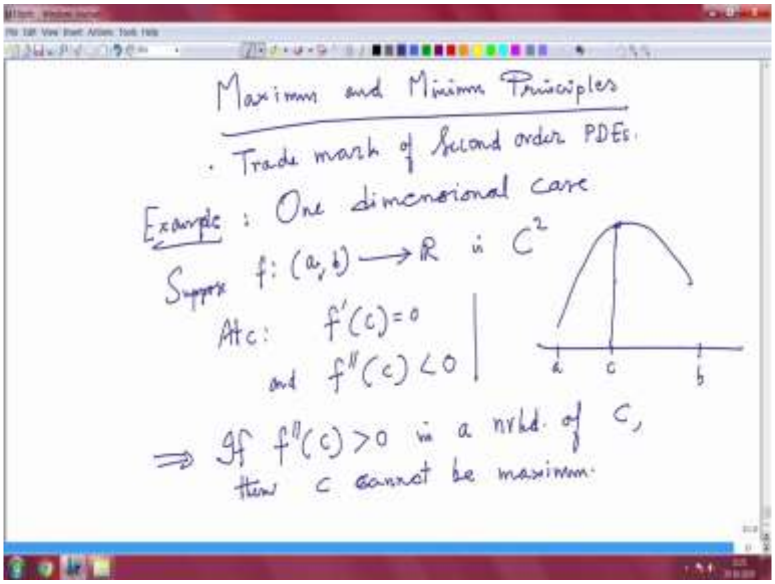

Now with this let me go to another important thing what we call it maximum principles, maximum, another important thing, maximum and minimum principles. This is basically a trademark of second order differentiation, though you may be able to prove some results for higher order, second order PDE. Why this I say that it is a trademark, so let us look at an example, one dimension case, example, one-dimensional case.

Suppose you have a function f defined in f, defined in an interval a b and 2 r, suppose, is C2 that means twice so you have an interval here, here and suppose f prime, look at the function something like that. What can you tell about this point? So this is your a b and this is your point c that is maximum point. At c you know that f prime of c equal to 0 but a frame of c can be 0 even in the minimum point and at the maximum point you get a double prime of c is negative.

So what does this conclusion tells you? If f double prime of c is positive in a neighborhood of c then c cannot be a maximum, c cannot be a maximum. This is one of the motivation which you get it. So whenever you find that f double prime of c is positive in some certain region in the neighborhood of c, then you can say that that function cannot have maximum there at that point.

So in the general elliptic operators this f double prime that is what I said, it is a trademark of second order. This replaces by second order operators so what kind of operators you can derive such a maximum principle. So what you will see in this class, not about general elliptic operators, we will derive such maximum principles soon for the Laplacian operators so that Laplacian replaces the second derivative or other elliptic operators.

So you can refer other books like other than our book, you can also get into other PDE books like Gilbarg and Trudinger to see various maximum principles in the context of second order, context of maximum principles.

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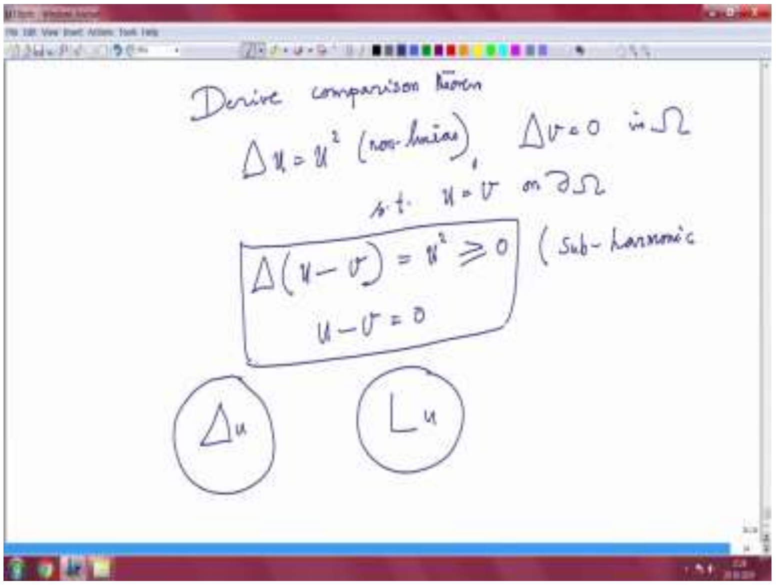

So we want to derive. This also has had sometimes this can use, you can derive comparison theorems so certain operators, certain theorem, say suppose you have an equation of this form. This is a nonlinear operator and you can compare results, compare the solutions between these two. If Laplacian equal to such that, suppose we know that u equal to v on the boundary of a domain, suppose you know that.

This is in omega, you know that, and then if you subtract it, you get Laplacian of u minus v is equal to u square. You will get it as, this you immediately see that this is sub harmonic which we are going to derive now equation and u minus v equal to 0. So you will be able to compare solutions between a problem which may be little difficult but then other problem will be easy. You will be able to do some comparison results.

So our aim is to derive maximum principles for the operator, Laplacian, but you can have more general operator which we will not be doing it here. Maybe I will give an example if I have enough time.

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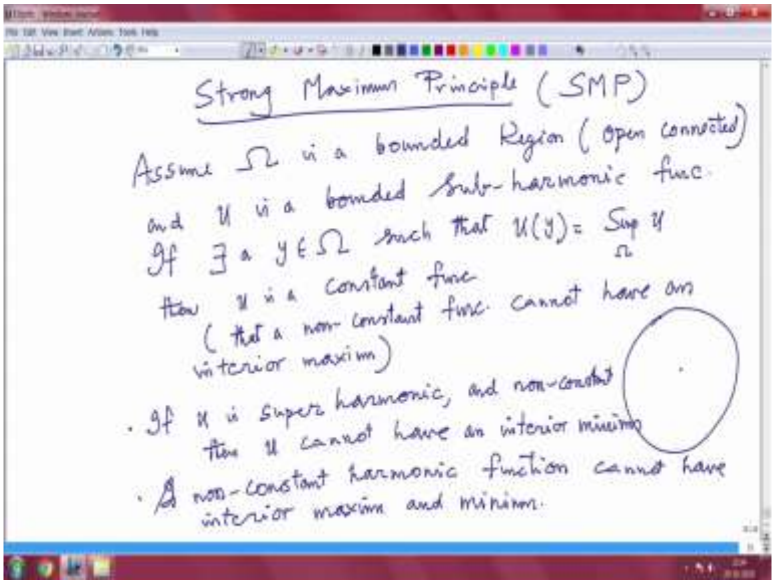

So let us derive a strong maximum principle. The proof is not difficult but the result is very very important and you also see an application of the mean value theorem. You call it SMP, you may refer it that way, SMP. Assume omega is a bounded region that means open connector. Otherwise you can apply to each connector compound and u is a bounded sub harmonic function.

If there exists a point y in omega such that u of y is equal to the maximum of u or supremum, so let me even write supremum of u in an omega, then the conclusion is, then u is a constant function, it is a constant function. You see, that means that is in essence what it says that a non constant function cannot have an interior maximum.

Try to understand the theorem very clearly. That means if you have an interior point that means you have a domain like this and if you have your y and maximum is obtained or the supremum is obtained in an interior point, then that function has to be a constant. A non constant function cannot have an interior maximum for a bounded sub harmonic function.

So if u is super harmonic, the other way result, if u is super harmonic and non constant then u cannot have an interior minimum. And the third result, a non constant harmonic function cannot have interior maximum and minimum. So it is enough to prove the first part because once you prove for a sub harmonic function that and it is a non constant sub harmonic function cannot have an interior maximum for if u is super harmonic, you apply the same result for minus u.

So minus u cannot have an interior maximum. That means u cannot have an interior minimum. And if it is a non constant harmonic function, you apply the results for both u and v so it cannot have either a maximum or a minimum.

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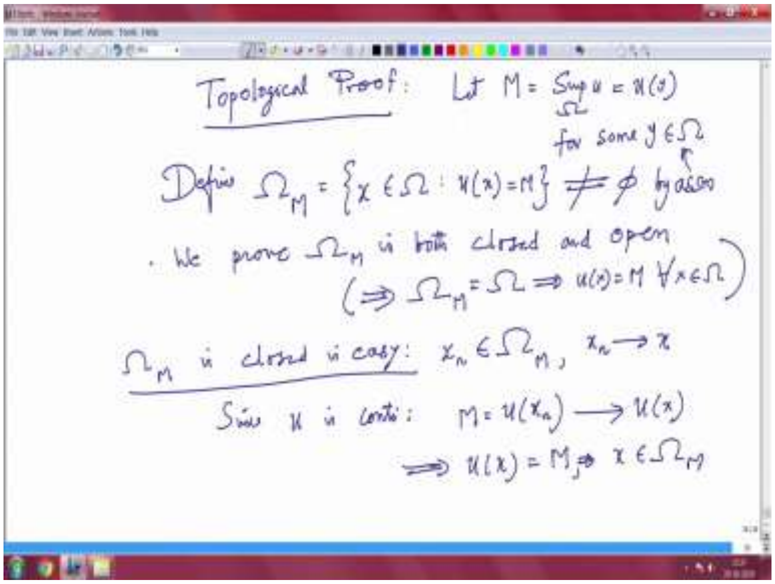

So let me try to give a proof based on a topological proof basically. So let m is equal to supremum of omega, supremum over omega of u and that is given to be u y for some y. There may be many y, y in omega. So you define omega m is, you collect all the points, set of all points y in omega such that u of y, set of all points x, y is a given point, set of all points x in omega such that u of x is equal to m which is non-empty by the given assumption, equal to m p by assumption.

So we will prove that, we prove omega m is both closed and open. So once you prove this, this will imply if it is both closed and open, is non-empty, omega m is equal to m, omega itself because if it is both closed and open, either it has to be empty or it has to be full domain omega and that implies u x equal to m for all x in omega. That is it. So it is proved. Omega m is closed is easy so we will prove first that, so you chose x n in omega n omega m and x n convert just to x, but then u is continuous.

Since u is continuous what do you get it, x n is in omega m so that m is equal to u x n and by continuity u x n convert just to u of x. that implies u of x is equal to m that implies x is in omega m. So omega m is closed. That is a true result by the continuity of your, so this is something like a kind of surface, immediately you see that.

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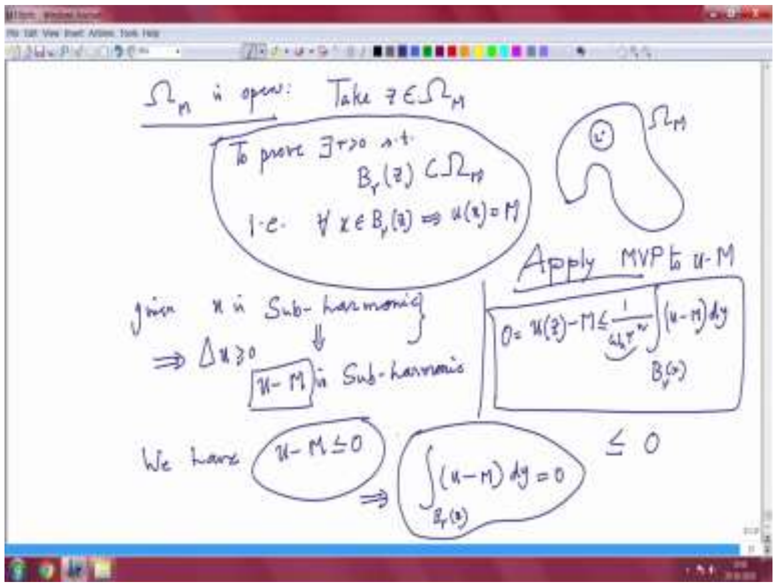

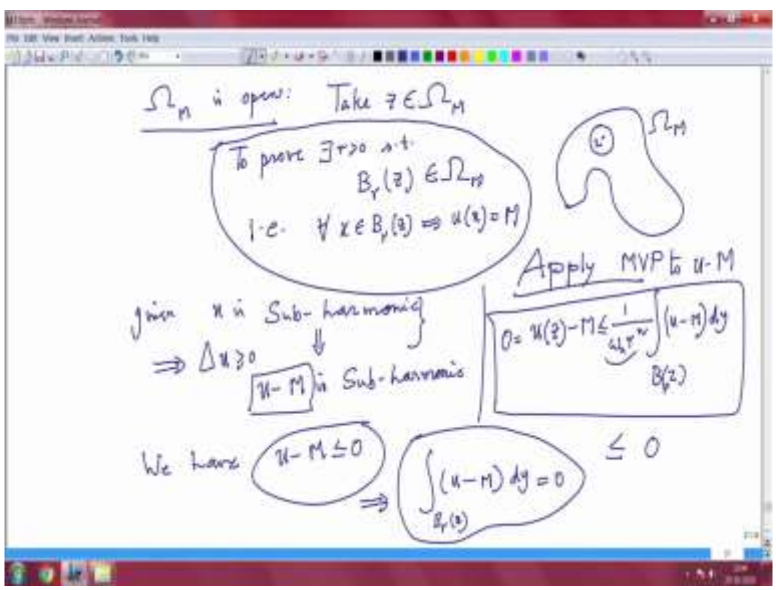

So let us prove omega m is open. That is a bit involved and you have to prove this. So you have a point, so you want to take a point, so what do you want to prove it, you have to take a point z in omega m. So you do not know omega so just, suppose this is your omega m. This is not omega, careful, so omega m will not be open you want to prove it.

You want to choose a point here; you want to choose that there is a neighborhood. You want to prove there exists r positive such that B r of z is in omega m. You have to prove it. That is you have to prove for all x in B r of x z implies u of x equal to m. So you have to prove that. So you have to prove this result. So this is what you want to prove it basically.

So you want to prove that there is, if you start with a point here, you want to prove that the entire neighborhood belongs to omega m. So what is given? You are given that u is sub harmonic. Given u is sub harmonic that implies Laplacian of u is greater than or equal to 0 but then this also implies since m is constant, u minus m is also sub harmonic because m is constant, this is also sub harmonic because m is a constant.

So you prove, you can apply mean value property, inequality or mean value property. You apply mean value property to u minus m. M is constant so you get at the point z, use z minus m because m is a constant, minus m, this will be, I can integrate for any ball, okay. So in particular again radius r, so I have 1 by omega n, any r power n, integral over b r of x, this is true because it is sub harmonic everywhere. So you have your in a ball u of y, dy.

Not u of y, you are applying these two, u minus m so I will have integral over u of, u minus m, dy. So this is the result we get it immediately by mean value theorem. So what do you get it? Integral of, so but what is u minus m? M is the maximum thing so you have to conclude something from this inequality, u, m is a maximum. We have u minus m is always negative.

So u minus m here is negative so this is less than or equal to 0, so you have both ways 0 so that means this implies integral of, this is a positive quantity so this is greater than or equal to 0. Yeah sorry, here we have to write, because it is only sub harmonic, so you have to write inequality. So if 0 less than or equal to this, because it is not a harmonic function, it is a sub harmonic function. Otherwise it is fine. So you have this one equal to 0, so that implies u minus m dy over ball of radius r equal to 0. So how do you prove, this integral is negative.

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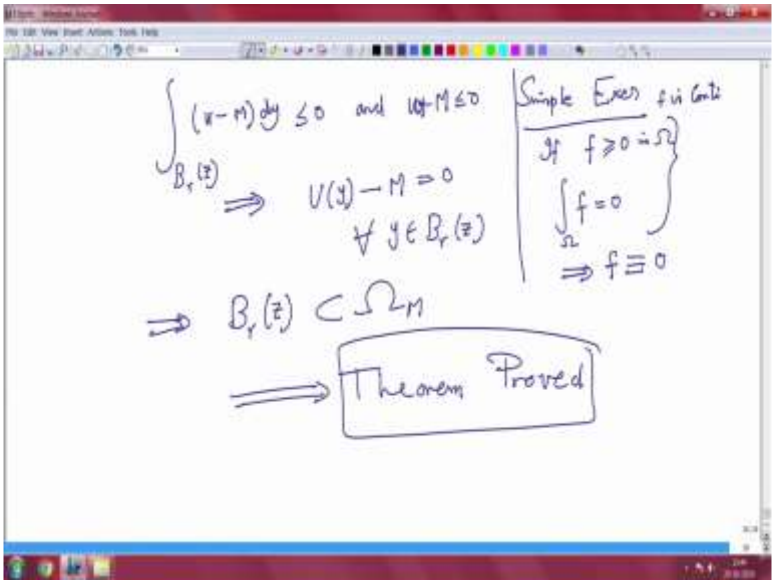

So that is a one more step. So you have an integral so here you use the result. So may be if you are not familiar, prove this, small simple exercise. Simple exercise tells you that if f presents a sign, say f is greater than or equal to 0 and integral of f equal to 0 on domain omega, in omega, these two will imply f identically 0 so assume f is continuous. So that implies f is identically 0. That is all we are assuming.

So we have integral of, similarly for negative, u minus m, dy, over b r of x is less than or equal to 0 and u minus m less than or equal to 0 for all y, u of y minus m. So that implies your u of y minus m equal to 0 for all y in b r of x. That implies b r of x contained in omega m. That is what we want to prove it. This is what you want to prove it, b r of z, so you apply it for this. So let me correctly write it, so this is for z, so you have taken that point z.

So this is for the point z so instead of x you have taken this point, does not matter you can take x instead of z so that proves the theorem, theorem proved. So what we will do we will make some more, this is a strong maximum principle and with this strong maximum principle we stop for the time being this class and then from here we can derive what is called a weak maximum principle and then I will make some more comments and then we will prove the uniqueness of our boundary value problem, not existence.

Existence you may not see except in special cases, the general existence theory will not be covered in this course because that needs little more terminologies and more time. So we immediately prove is the, so you see the mean value theorem, maximum principles, these are all properties we have derived assuming that there is a solution, nice solutions we have derived and from there we conclude the uniqueness of the theorem. Thank you.