First Course on Partial Differential Equations – 1
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Lecture 15
Partial Differential Equations – 1

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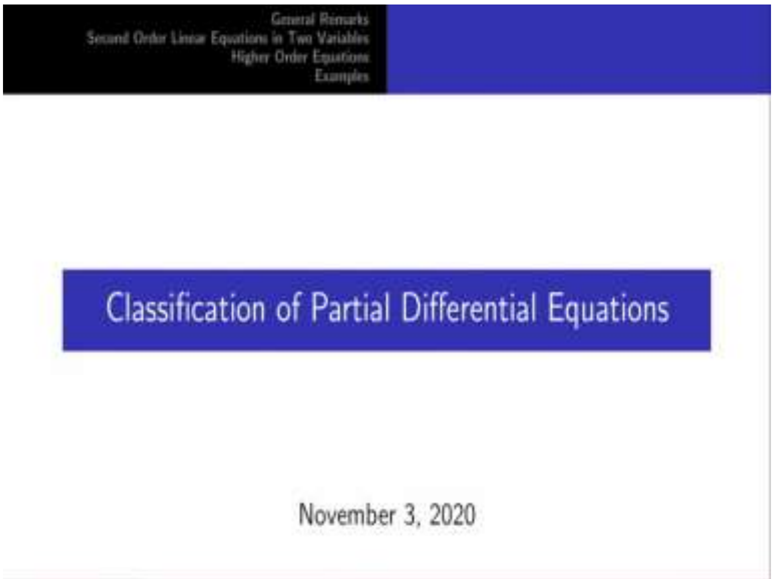

Today, we will start the topic of classification of partial differential equations. There will be a few lectures on this topic, and we concentrate essentially on second order linear partial differential equations for the classification and you will see why this restriction as we go on.

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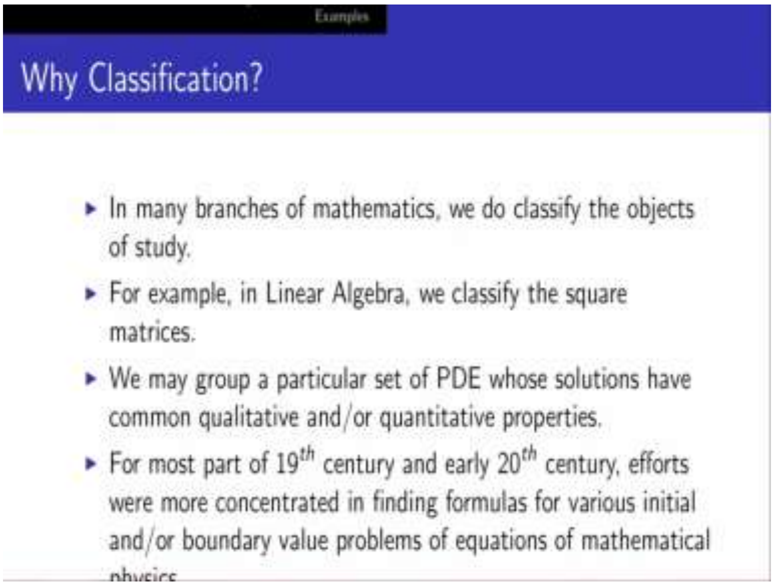

So, first of all, we start with the basic question, why classification? So, in many branches of mathematics, we do classify objects of study. For example, in linear algebra, we classify square matrices into class of symmetric matrices, orthogonal matrices, and positive definite matrices. Why do we do that? So, in order to -- so, this we may group a particular setup PDE, whose solutions have common qualitative or quantitative properties.

So, that will recognize given a PDE into which group it falls. For most of the 19th century and early 20th century, efforts were more concentrated in finding formulas for various initial and our boundary value problems of equations of mathematical physics. And that is the reason in that period people essentially concentrated on three types of equation. The first one is wave equation.

So, describing vibrations in -- small transversal vibrations of a string, the longitudinal vibrations of rod, electrical oscillations in a wire, etcetera. And this equation is a prototype of hyperbolic equations. So, we will define what is an hyperbolic equation as we go on.

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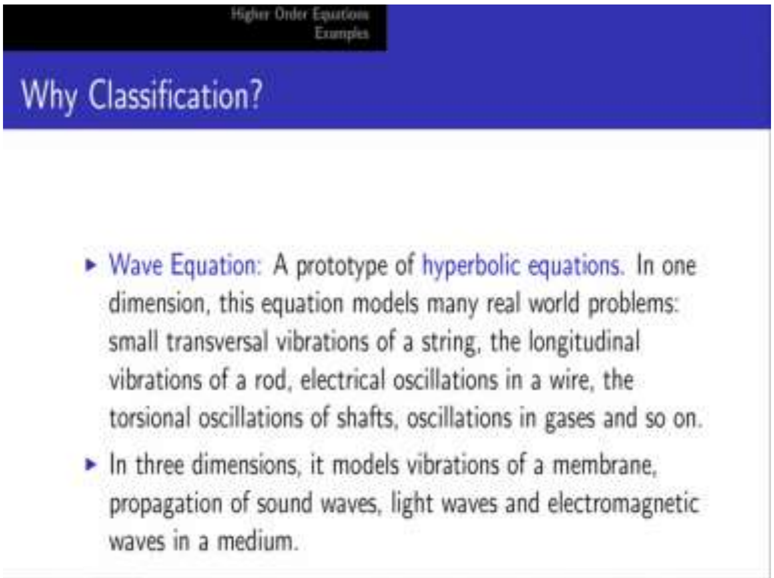

So, this typically the hyperbolic equations, they have qualitative properties connected with the solutions of the wave equation. So, this is in one dimension and it also has in three dimension, it models vibrations of membrane, propagation of sound waves, light waves, and electromagnetic waves in a medium.

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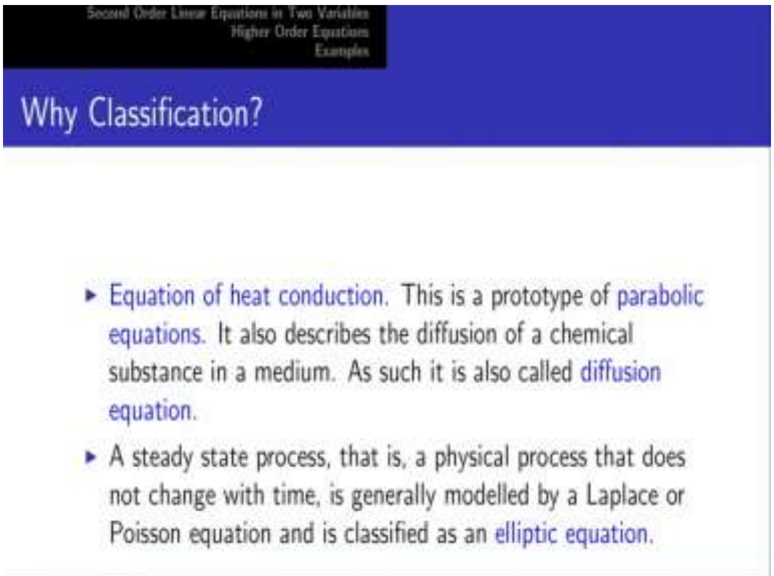

And then comes the equation of heat conduction and this is a prototype of parabolic equations. So, this is another classification in those. So, one of the again physical phenomena of heat conduction, so they were interested in that. So, this was studied in great detail.

So, it also describes diffusion of a chemical substance in a medium as such it is also called diffusion equation. So, heat equation or diffusion equation. And the last one a steady state process, physical process, there is a physical process that does not change with time is generally modelled by a Laplace or Python equation and this is classified as elliptic equation.

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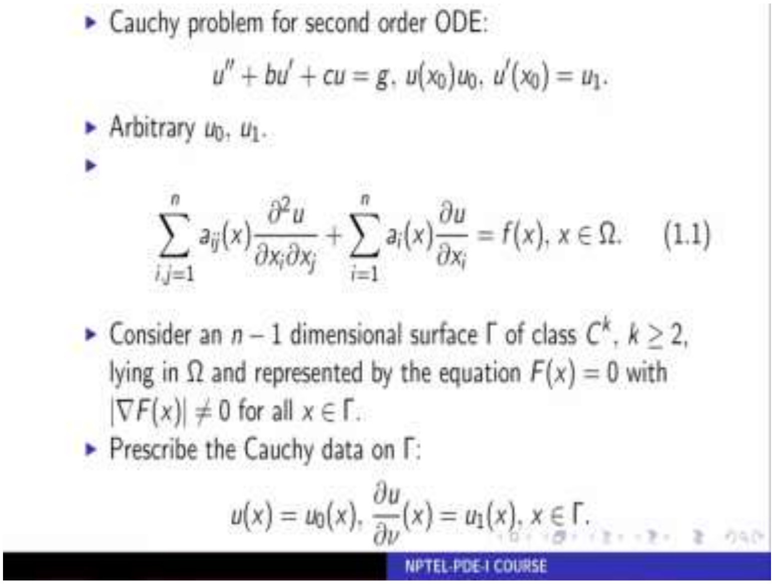

We have one more mathematical reason for doing the classification. Given a differential equation, whether ordinary differential equation or a partial differential equation. A basic problem associated with an ODE or PDE is the Cauchy problem or an initial value problem.

So, for a second order ODE, u double prime bu prime plus cu equal to g, in an interval on the real line. So, we prescribe two initial conditions ux0 equal to u0, u equal to is missing there, equal to u0 and u prime x0 equal to u1.

So, initial position and initial velocity, if you interpret in Newtonian mechanics, so this describes the dynamics of a particle. So, if you give initial position and initial velocity, we should be able to tell the dynamics of that particle. And in case of ODE, there is no restriction on u0 and u1, they are arbitrary.

So, such a scenario changes when we come to a PDE. For example, here consider a linear second order equation in some domain in Rn. So, you can just take R2, for example. And what is the Cauchy problem associated with this equation? Cauchy problem, you can give it on a hypersurface.

For example, in R2, this is nothing but a smooth curve. So, in higher dimensions it is an n minus 1 dimensional surface and that is called a hyper surface. And so, suppose it is represented by an equation of the from fx equal to 0 with grad effects not zero for all x on their surface gamma. And this condition just indicates that there is a well-defined normal to the surface gamma at every point and it is smooth. That is why you assume this gamma to be of class Ck.

And on this surface, we prescribe the initial data or Cauchy data it is called for the equation 1.1. And so, this is the prescribed the Cauchy data on gamma, so you give essential position and then you give the normal derivative. So, this del u by del u denotes the normal derivative of u on gamma, and these two are prescribed. So, similar to the ODE problem.

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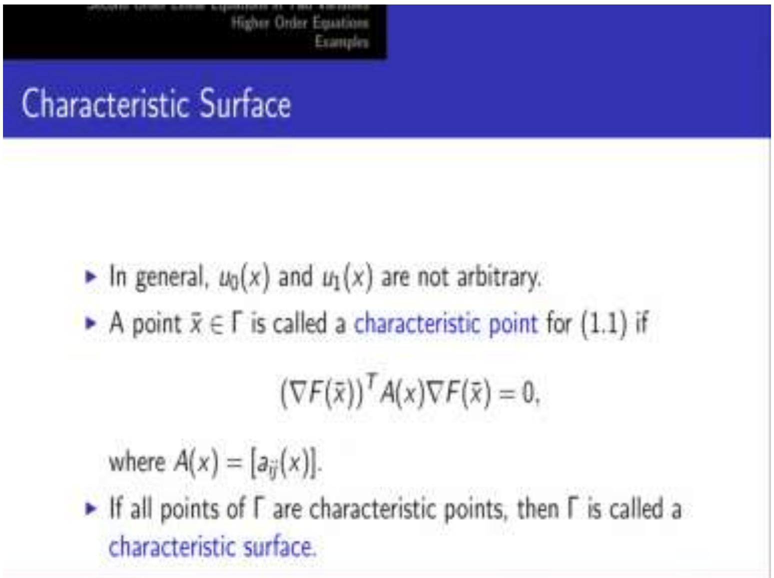

So, two initial conditions for the ODE are given. So similarly here it is a second order PDE, so we are giving two conditions. And unlike ODE, what we observe is that, in general this u0 and u1, x are not arbitrary. So, this requires a great deal of computation, which I am avoiding here. So, that is the difference. What makes this difference is the fact that -- so this we did not face in the case of ODE, but in case of PDE, we do face this.

And so, the main reason for that is whether gamma is a characteristic surface or not, characteristic surface with respect to the given PDE. So, here is the definition a point x bar and the gamma is called a characteristic point for the equation 1.1, so important. So, that you are connecting now, the initial surface gamma and the PDE, that is important.

If this grad fx, t, superscript t denotes transpose of a matrix, and A is coming from the PDE, that matrix is -- so, it is coming from the coefficient CIJ. So, which I denote by the so it is a variable matrix. So, it is -- if A is our constant, then it is a constant matrix, otherwise it is a variable content. So, there should be x bar here, sorry for that.

So, at that point, so if this happens is called a characteristic point. And if each point of the surface gamma is a characteristic point then gamma is called a characteristic surface. So, if gamma is a characteristic surface, then the Cauchy problem is called a characteristic Cauchy problem. And if gamma is non-characteristic surface, then the Cauchy problem is called non-characteristic surface, non-characteristic Cauchy problem.

And generally, non-characteristic Cauchy problems, they have solution that can be shown. And in case of already first order PDE, we have seen this characteristic in some other form, namely transversality and other conditions. So, it is similar to that. So, in general -- so, this -- that is the difference. So, and this definition of the characteristics obviously depends on the PDE with the coefficients and that is why you want to classify the PDE into different groups. So, this is the mathematical reasoning for studying this classification of PDE.

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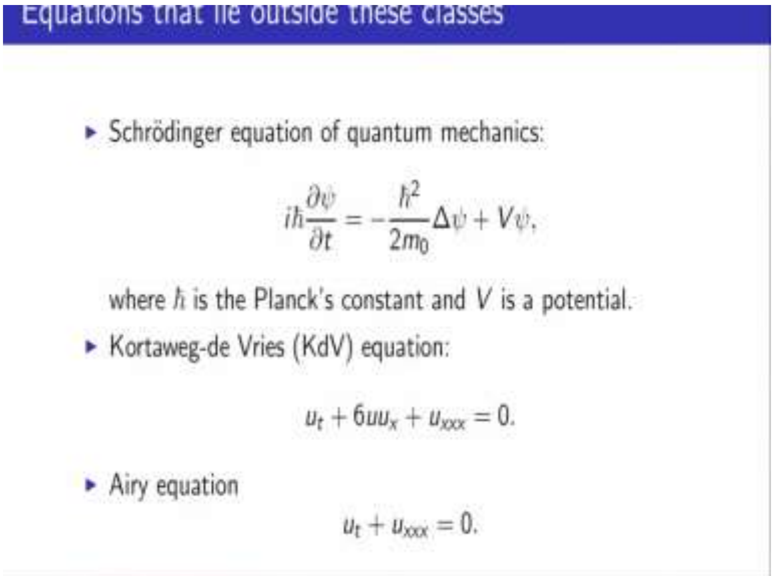

Having said about that, about the classification, there are many, many equations of importance that lie outside these classes. What are these classes, namely hyperbolic, parabolic, and elliptic. So, I just list a few of them, there are plenty of equations. So, one is this Schrodinger equation from quantum mechanics it is written here.

So, it looks closer to the heat equation, but qualitatively, if solutions are more like wave equation. And also you observe that, this i, i is square root of minus one. So, we have to deal with as complex functions as solutions. So, in case of heat equation, wave equation, and the Laplace equation, they were all real valued functions. But in case of Schrodinger equation, we have to deal with complex functions as solutions.

So, the next one is again this is very much studied in the literature, commonly called KdV equation, this Korteweg–De Vries equation. It is also an – it is a non-linear equation of third order. So, uxxx. So, here by looking at the nature of the solution, it is classified as this pursue equation as opposed to diffusion equation.

So, the linear equation of KdV, which existed much earlier is called Airy equation. So, just you remove the non-linear part in the KdV equation, you get the Airy equation. And then there are equations from fluid dynamics, Navier Stokes equations. So, that is a system of equations. So, that also does not easily fit into this classification and there are many more. So, you can just see the literature.

So the in what I am trying to say here is this classification is incomplete. So, there is – it is not that all PDEs will fit into one of these three groups; there are many important PDEs, which lay outside these three groups. And in the modern theory of the subject, especially after the 1950s, so there is quite a bit understanding of partial differential equations. And now we see the classification as hypo elliptic and non-hypo elliptic. So, even to make a definition of that, we have to develop many tools, so which is not possible in this first course on PDE.

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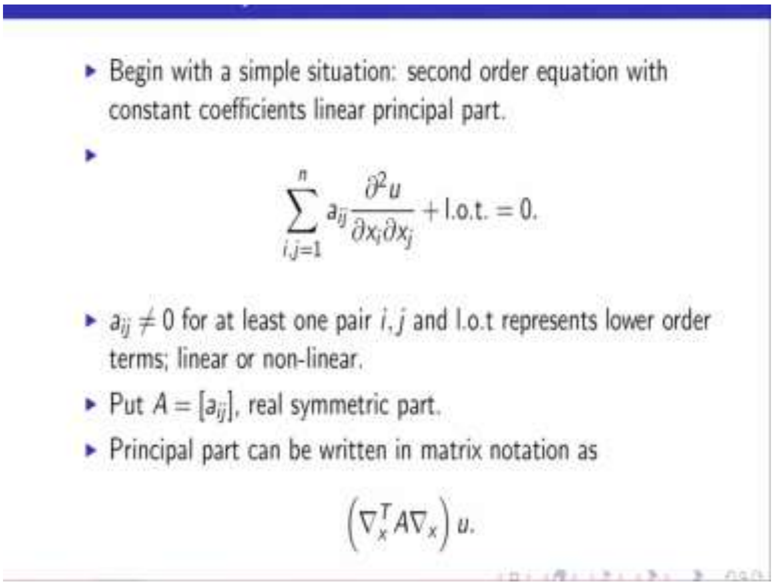

So, we will do some basic thing in this course. So, begin with a simple situation just to understand what did this classification and how we are going to do this classification. So, let us take the simplest example. So, a second order linear PDE. So, as far as the classification is concerned, only the second order derivatives matter.

So, that is why return only an second order the highlighted and then l.o.t. means is lower order terms, just concentrate on the highest order terms. To qualify it has a second order equation, so at least one coefficient aij should be not 0, that is what the written is. Otherwise, if your all aij are 0, then we are getting first order equations. So, the lower order terms whether linear or non-linear, it does not matter as far as the classification is concerned.

So, here -- so, I denote by A, capital A, the matrix of the coefficients and that is a real symmetric matrix. That A is the real symmetric matrix. And so, this part aij del square u by del xi, del xj is called principal part of the equation, and that is written in the matrix notation as del x transpose a, del xu, del x denote the gradient vector, and x signifies we are differentiating with respect to x variable.

So, we are considering the equation in some open set in Rn, so that we denote the points in that open set x. Because then we are going to change the variable x to some other thing. So, then we write it as del of that variable. So, that x signifies the differentiation is with respect to x.

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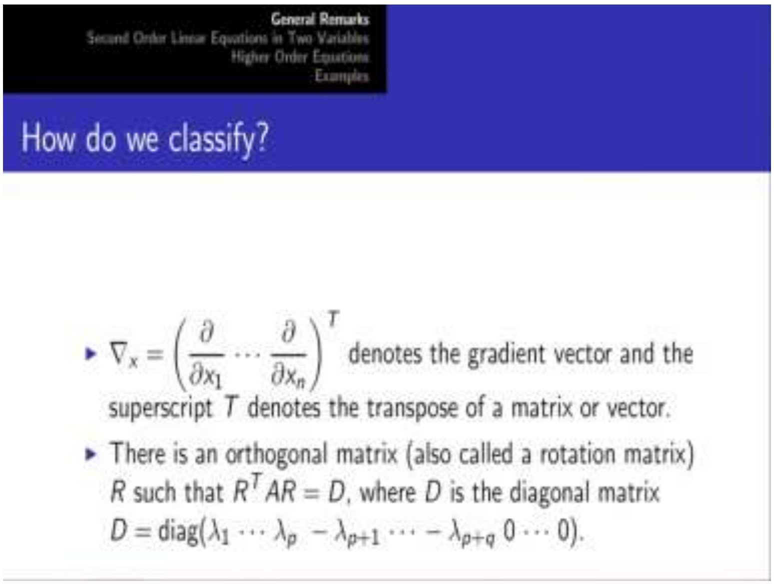

So, this written it here, so it is the gradient vector and superscript denotes the transpose of a matrix are a vector. From linear algebra, we learned that , since A is a symmetric matrix, there is an orthogonal matrix it is also called a rotation matrix, R such that R transpose A, R equal to D. So, R -- since R is an orthogonal matrix, R transport is also equal to R inverse. So, that is the knowledge we take from the linear algebra.

And indeed, so in a essence what it says is a symmetric matrix is diagonalizable and we can arrange the diagonal elements of this matrix D in the following way. So, lambda1, lambda2, lambda p, minus lambda p plus 1, minus lambda p plus q, and rest are 0. So, here we are collecting the positive Eigen values of A negative Eigen values of A and zero Eigen values. Of course, A can have all the positive Eigen values, all the negative Eigen values.

So, it cannot have all the zero Eigen value in this case, because A is not a 0 metrics. So, that is what we are writing now.

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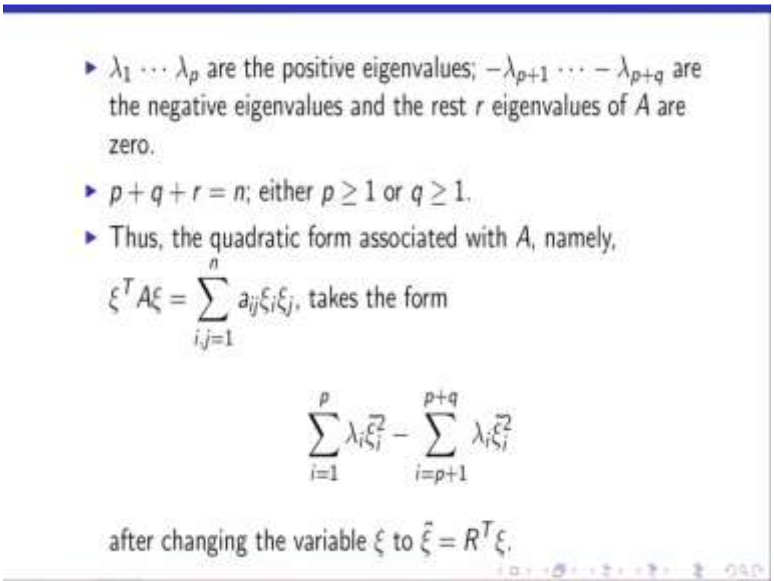

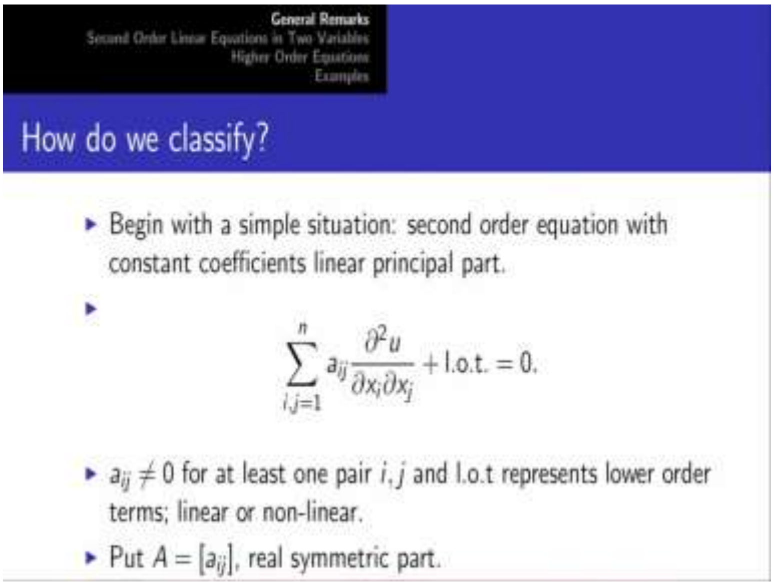

So, lambda1, lambda2, lambda p are the positive Eigenvalues of A, minus lambda p1, etcetera. are negative Eigenvalues. So, there are p positive Eigenvalues, then q negative Eigenvalues and R 0 Eigenvalues, for total numbers should be n, so p plus q plus r equal to n, and since A is non-zero, so it has at least one non-zero Eigenvalue.

So, that means, either p is bigger than equal to 1 or q is bigger than equal to 1. Thus the quadratic form associated with A, so this is also quadratic form associated with the given PDE. So, again let me just go back. So, in this summation Aij del square u divided by del xi, del xj, you replace that del square u by del xi, del xj by ji, jj.

And that is how you get the quadratic form associated with the matrix A. And matrix A comes from the PDE, so this is also a quadratic form associated with the PDE itself. And then you change the variable, you change from j to j tilde; j tilde is equal to r transpose j. So, r is coming – r is the matrix that diagonalizes edges a, r is the rotation matrix.

So, that using the diagonal process. So, you just get that quadratic form takes the diagonal form. So, there are no ji tilde, jj tilde. So, it is purely a diagonal form, lambda ij, i tilde square corresponding to positive Eigenvalues and then we have correspond a term, corresponding to negative Eigenvalues. So, of course, the rest are zero, so you just they will not appear.

And in essence, what we are going to do for the classification again going back to the PDE we want to somehow get rid of the mix derivatives, del square u by del xi, del xj, i different from J by finding suitable change of variables, we want to make it just a diagonal term. So, fewer second order derivatives, del square u by del xi square.

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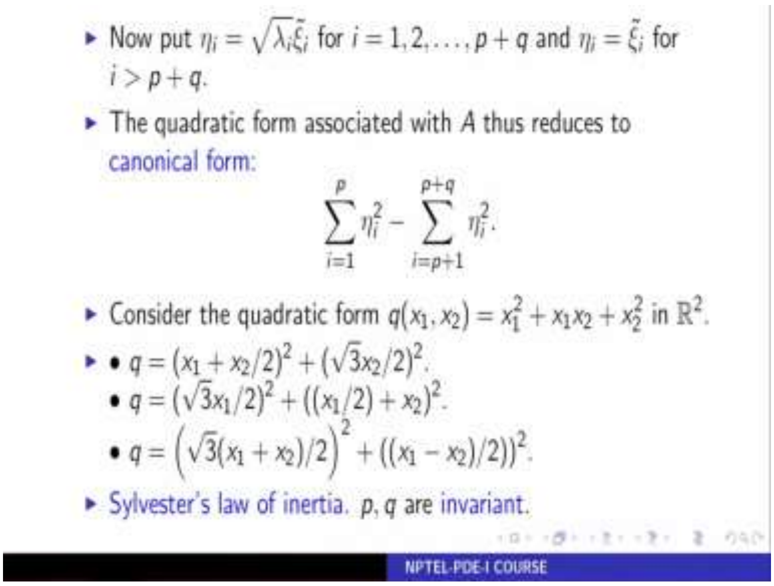

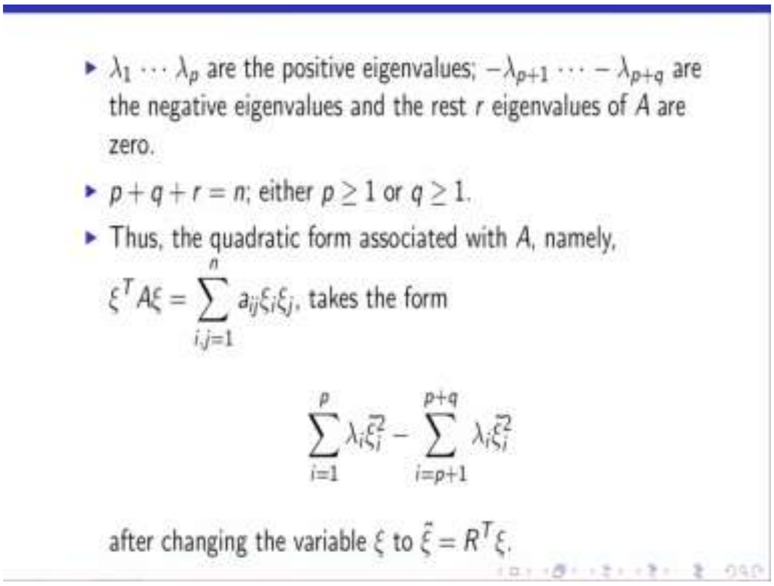

Similar to this diagonalization of the quadratic form associated with the matrix A. And we can do one more transformation the put eta, i equal to square root of lambda i ji tilde for i equal to 1, 2p plus q. So, wherever you see that, so I want to observe this lambda i into ji tilde that is what I am doing and that gives me this only plus ones and minus ones, nothing more. And this is called the canonical form associated with A or it could be quadratic form associated with A.

So, we get only some diagonal terms and coefficients are also either plus 1 or minus 1. One difficulty, immediate difficulty we see is, this obtaining canonical form, we have used the diagonalization of A and arrived that this canonical form, but that is not the only way one can arrive at a canonical form. So, there are many different ways. So, here let me just give you one example in R2.

So, we consider this quadratic form q, x1, x2 equal to x1 square plus x1, x2 plus x2 square in R2. So, I can write this quadratic form. So, there is x1, x2, so I want to remove that. So, in three different ways, in fact, you can write in many more ways, but I have just written here three different ways how you can write q.

So, something square, something square, something square, something square. So, I can use all these three different chains of variables in order to write q in its canonical form, or so canonical form is not unique, but what is unique is and that is again from the linear algebra, the number of positive Eigenvalues p, the number of negative Eigenvalues q, they are invariant and this is called Sylvester’s law of inertia.

So, whatever change of variables you use to reduce a given quadratic form to its canonical form, the number of p and q will not change and that is an important observation. And based on these invariants, p and q, we are going to classify the PDE. So, now, you immediately connect this observation from linear algebra to our PDE.

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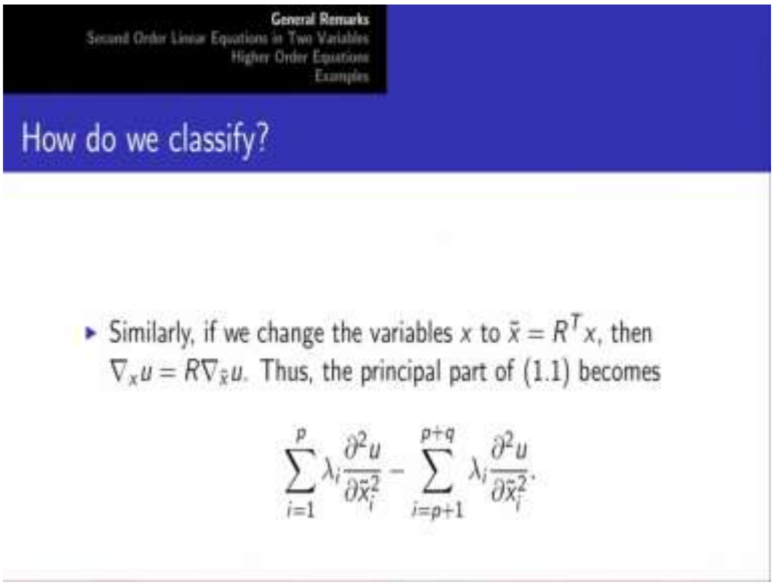

So, now, just again use that rotation matrix and you change the variables x to x tilde and then, so this gradient vector of u with respect to x variable changes to gradient of u with respect to x tilde variable by that rotation matrix. And that you put in the equation back and you see immediately that we obtain the principal part in diagonal form. Still there are lambda i's. So, we will also remove them by one more change of variable.

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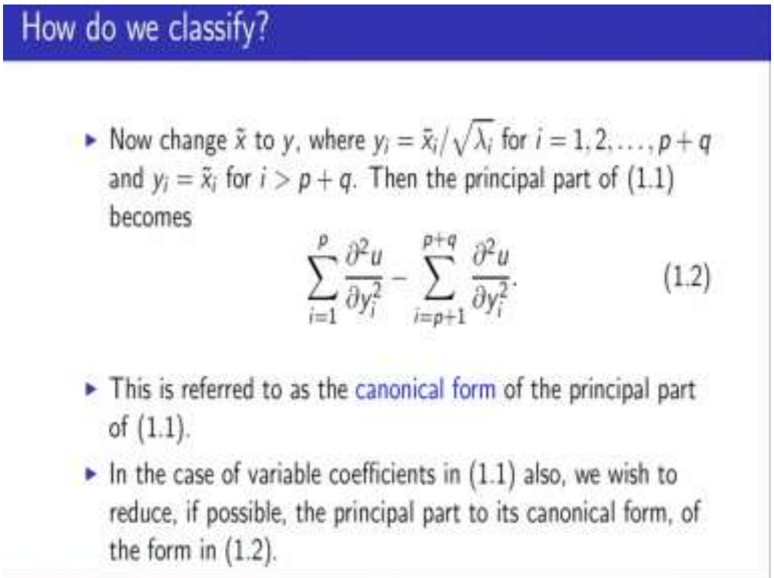

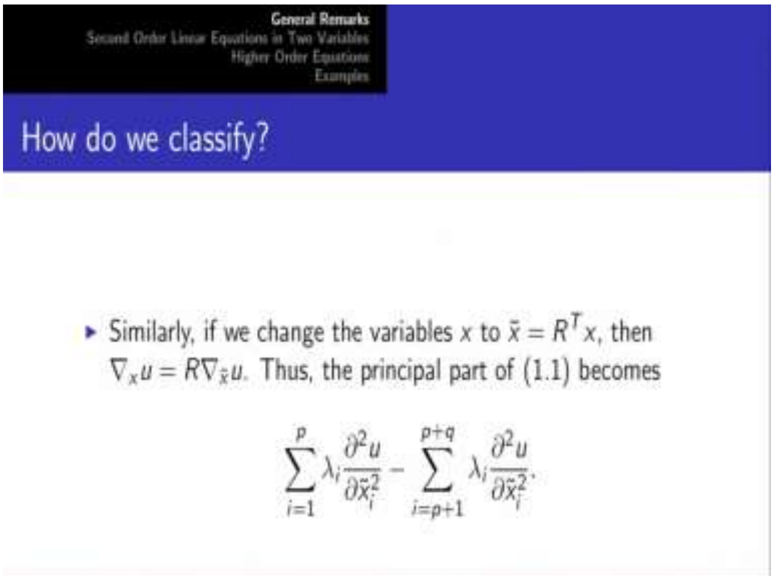

And now, you change x tilde to y. So, again just you want to absorb this lambda i in xi tilde and that is very easy. So, you put yi equal to xi tilde by root lambda i, and that converts the principal part into a diagonal part del i square u by del y square and with coefficients only plus and minus 1.

So, you just look at equation 1.2, you see that the coefficients are only -- there are p plus ones and q minus ones. And this is referred to as canonical form of the principle part. So, we are not bothered at this stage about the lower order terms, the first order terms. So, we are just concentrated on the second order terms.

So, before defining the classification terms, so in case of variable coefficients, so when aij are also functions of x, we wish to reduce the principal part into canonical form of the form 1.2, if possible, that is the big question whether we can do it or not. So, just that will come a little later.

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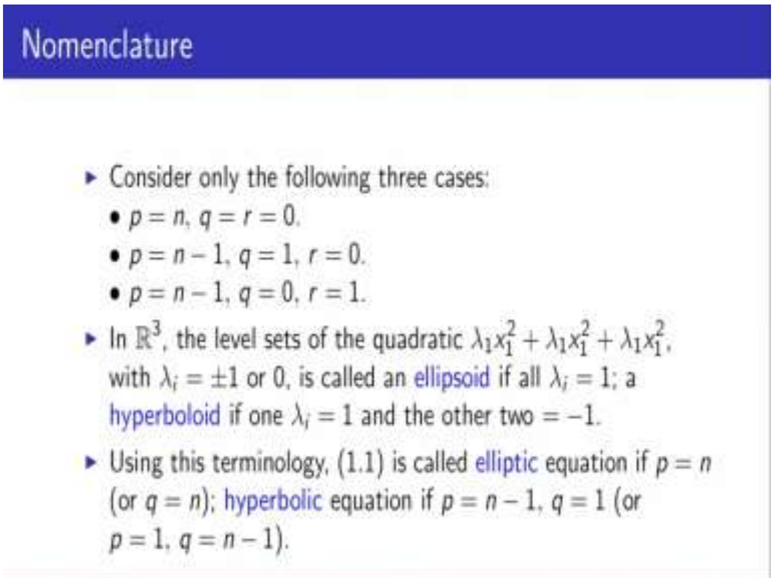

So, let us now -- so here only three cases are considered, namely p equal to n, in that case q equal to r equal to zero. And p equal to n minus 1, q equal to 1 and r equal to 0. p equal to n minus 1, q equal to 0 and r equal to 1.

So, in the first case, all the Eigenvalues of A are positive, and in the second case, n minus 1 Eigenvalues are positive and 1 negative Eigenvalue. And in the last one, there are n minus 1 positive Eigenvalues and 1, 0 Eigenvalues. So, depending on this cases, we call the equation by a suitable name.

Again, the name nomenclature comes from geometry. So, let me just tell you that. So in R3, the level sets of the quadratic lambda 1 x1 square, lambda 2, x1 square, sorry for that. So, it is lambda 1 x1 square plus lambda 2 x2 square plus lambda 3 x3 square. So, if all the lambdas are positive, the level sets up that quadratic are ellipsoids. And if two of them are positive and one is negative, it is hyperbolic.

So, the same terminology we use for the classification of PDE. So, PDE 1 is called elliptic equation, if p equal to one or by changing the sign we can also assume q equal n. So, either of that, p equal to n or all our positive Eigenvalues or negative Eigenvalues. So, hyperbolic if p is equal to n minus 1 and q equal to 1 or again by changing the sign you can assume p equal to 1 and q equal to 1.

So, in these two classification, there are no zero Eigenvalues. So, in the first case, when p equal n or q equal 1, the quadratic form is called definite, and in the other case, it is called indefinite. So, in the definite case, either it will have all the time positive sign or negative sign. And in the second case, that hyperbolic case, it will have -- it takes both positive and negative. So, we will come back to the other case in the next class. Thank you.