An Introduction to Smooth Manifolds

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Orientation on Manifolds 2

Lecture 67

Hello, and welcome to this today's lecture.

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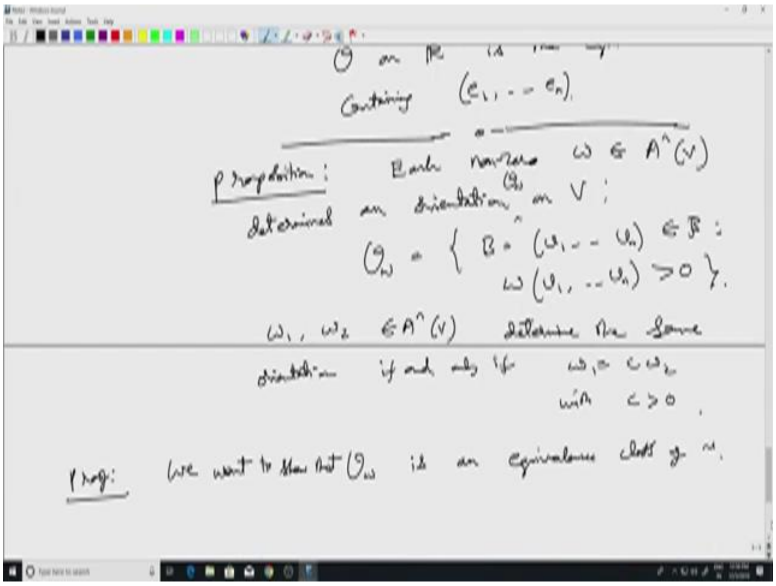

So let us resume at the point where I had stopped last time. So I wrote down this small proposition which says that if you are given a non-zero n form, then that gives an orientation on v. And we have a statement saying when they give the same orientation. So now let us see what really we have to prove here. Well, so what I want to say is that this, to say that this gives an orientation, is that I want to say that this is, we want to show that this thing here is an equivalence class of tilde.

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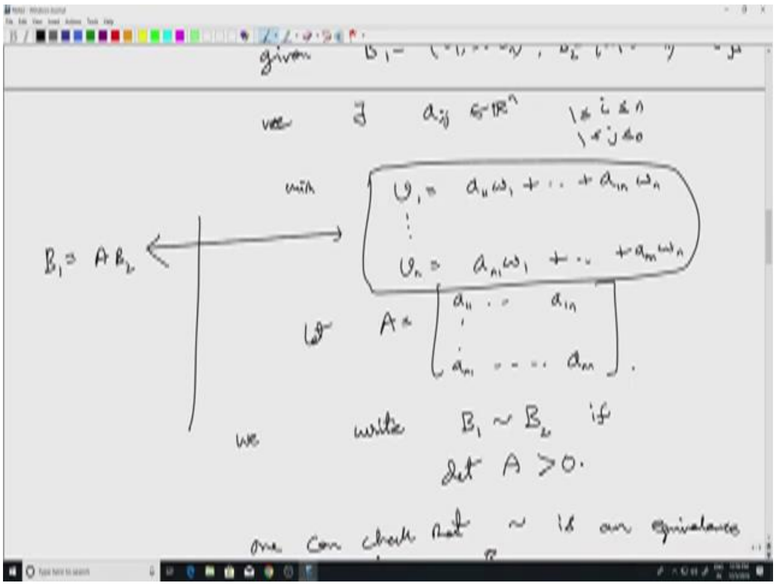

In other words, by the way this before I proceed this, the defining equation in terms of this matrix A, the change of basis matrix A, so let us write this, a shorthand notation for this set of equations. Let us write it as this, this B1 equals A B2. So, the meaning of this is precisely this.

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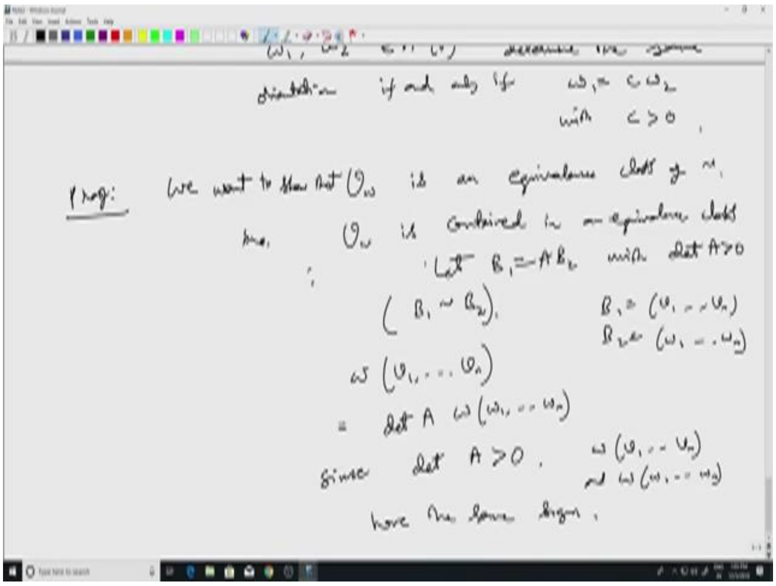

So using that notation, so we want to show that this O omega is an equivalence class of tilde So i.e. so first let us show that this O omega is contained in an equivalence class. So what does it mean to say it is contained in. If now the i.e. if B1 equals A B2 with determinant A positive, in another way of saying B1 equivalent B2. So anything, so I have taken an (equivalent) two things contained here, so is B1 equals A B2 with this. Well, instead of saying in other words, so let me just start like this.

Let B1 equals A B2 with determinant A positive. So let us see what this gives us. So this let us calculate v1 up to vn, as we have seen many times. In fact, one can just use the definition of the determinant to see what I am about to write down. So, this will be, if I write down v1 etcetera in terms of. So, v1 is with same notation as before, if I write down v1 in terms of the wi's, then this will be the same as determinant of the matrix A, then, this is not quite Ov, this does not make sense. So, what I want is omega, omega of this is equal to omega of w1, wn. So, we have done this several times in the previous lecture. So, therefore, and since that A is positive, Omega v1, vn and omega this have the same sign.

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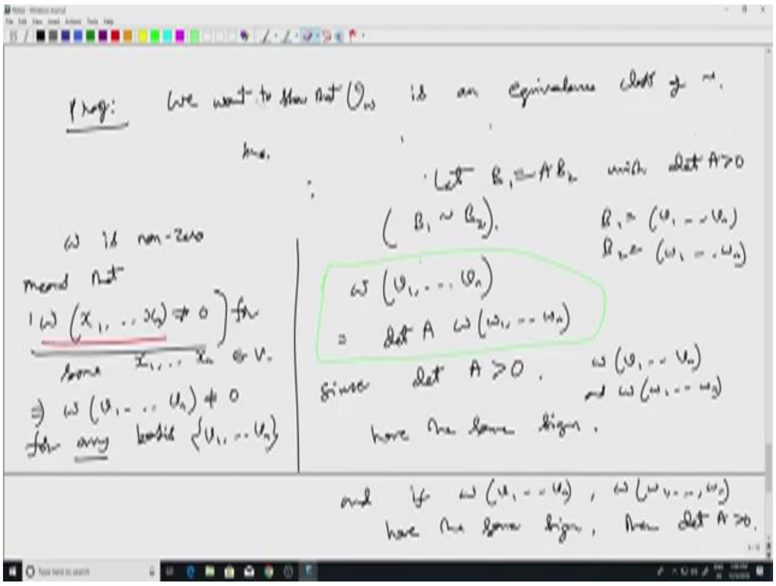

So, what does this tell us? And conversely as well if these 2 have the same sign then det A is positive. And if omega v1 have the same sign, then det A is positive. Also, I should have remarked that where have we used a hypothesis that omega is non-zero. Well, of course, here we are talking about sign, which does not make sense if it is 0. But what the hypothesis that omega is non-zero, implies, means that it is not the 0 form, it means that omega of x1, xn is not equal to 0 for some x1, xn in V, for some choice of Vectors. But this immediately implies that, if I write down these Vectors in terms of any basis, then this thing here, well the left hand side, since it is in n form, I will just get this will be a multiple of what it is for a basis Vector.

So implies that omega v1, vn is not equal to 0 for any basis v1 up to vn. So the assumption that the form is non-zero is just that it is not, there is some input, choice of input Vectors for which I do not get the 0 output. But that implies that if I plug in any basis Vectors, it will be non-zero, just by expanding this and using multi linearity and alternating properties of omega. Well, here, so this equation now shows that, this to say that omega v1, v2, vn and omega w1, w2, wn have the same signs, is the same thing as saying determinant is greater than 0. Well, that proves here what we want exactly. Actually, I do not need to say it is contained in and so on. At one go, we get it straight away.

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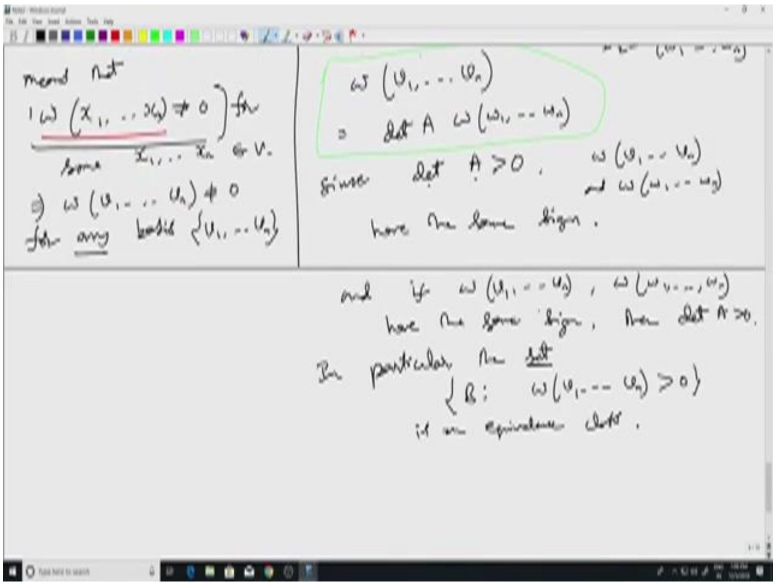

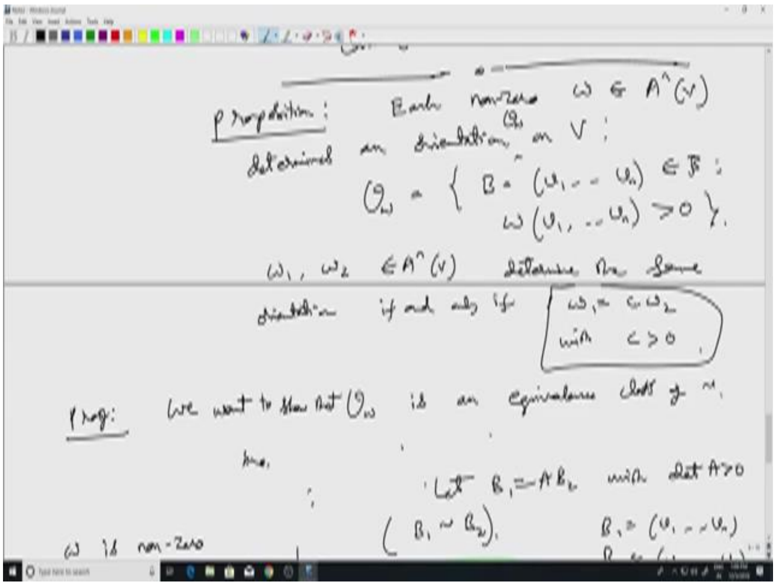

Simply because actually, I am interested in the case that this when this sign is actually positive, but I am leaving with in particular, the set of all basis such that omega is an equivalence class. Because, well if you have any 2 B's inside the set, B1 and B2 inside this particular set, then we have seen that the determinant is positive. So that means that this set is contained in an equivalence class. On the other hand, if there is something else in the equivalence class related to something in the set, again that new basis will have to have the same sign as omega, will also have to have positive omega. So that new basis will also have to belong to the set. So the set is exactly equal to the equivalence class. And it is clear that from the way it is defined that this condition holds that positive constant. This is after all, it is completely a tautology to say that the constant has to be positive to get the same equivalence class.

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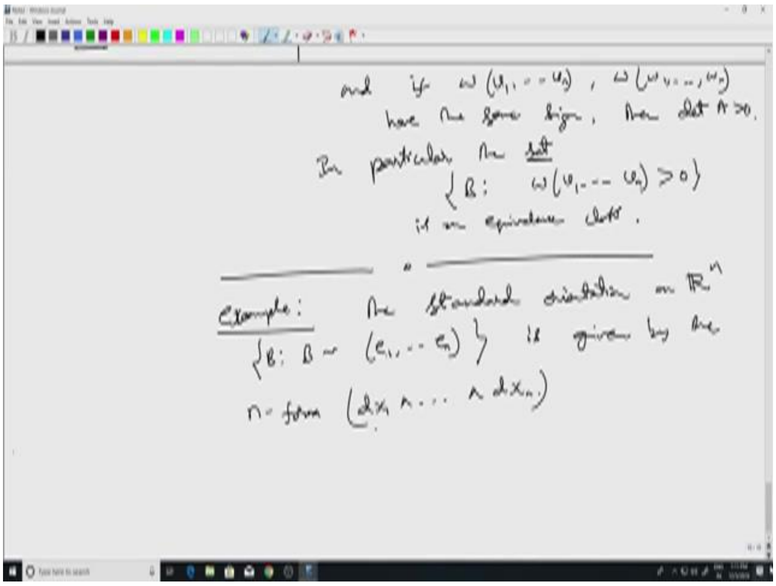

So, as an example, the standard orientation on Rn, so recall that this is the all basis B, which are equivalent to this basis, standard basis, written in an ordered manner, set of all B which are equivalent to this is given by the form dx1 wedge dxn, given by the n form dx1 wedge dxn, here we have a constant. So in other words, I just have to check that this particular form evaluated on e1 up to en is positive. And in fact, we know what it is exactly, it is this 1 forms are dual to these 1 forms. Therefore, I just get the value to be actually 1 and there is no permutation going on there is 1 to n here, 1 to n here. And an earlier lemma tells us that this n form evaluated on e1 up to en is exactly 1, which is positive. That is all I need to know, to say that this orientation is given by this form.

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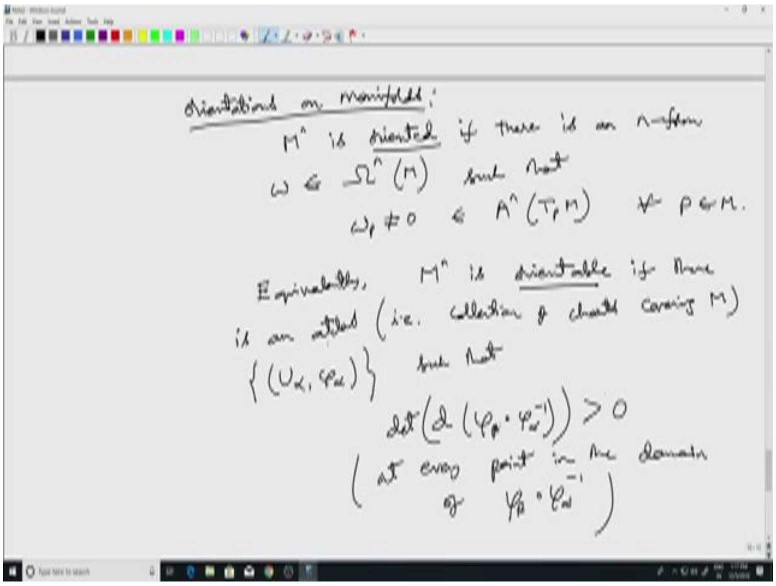

Now let us move on to manifolds. So there are 2 ways, one can, at least 2 ways one can think about in orientation. One is the, what we just discussed tells us that to have an orientation is the same thing as picking out n form, a non-zero n form. Actually, it is not quite that, because of this constant C business, it is picking out not a single form, but rather sort of half line of forms. So all multiples of some form, positive multiples of a given form is the same thing as having an orientation. But at least one thing is true that if I have a non-zero form on V, I will certainly get an orientation. So what one can try to do is, we say that M is oriented, if oriented or orientable is also another thing or when I say oriented, I already have a form in mind, M is oriented if there is an n form omega.

Actually, let me drop the P, omega and omega n, so the dimension of the manifold is n, of course, to present n form such that all I require that at each point, it is not the 0 form, omega P is not equal to the 0 form and An TPM and this is for all P in M, nowhere should it be the 0 form. This will do so as a definition because at on each tangent space I will get an, essentially I will get an orientation which amounts to saying, I will be able to say when, if you give me a basis of a tangent space at any point, I can say whether it is an oriented basis or not, because this omega gives rise to an orientation on each tangent space. Equivalently, here say that orientable as opposed to oriented if there is an atlas i.e. collection of charts, whose union is all of them covering M U alpha phi alpha, there is an atlas such that the derivative of the transition functions.

So whenever we have a transition function, so that will be phi, well phi beta composed with phi alpha inverse. This derivative, the transition functions will be diffeomorphisms between open sets in Rn, and the derivatives will be linear isomorphisms between Rn and linear isomorphisms of Rn. And so I look at the determinant of that, this should be positive at every point in the domain of this transition function. So at every point in the domain of eta composed with phi alpha inverse.

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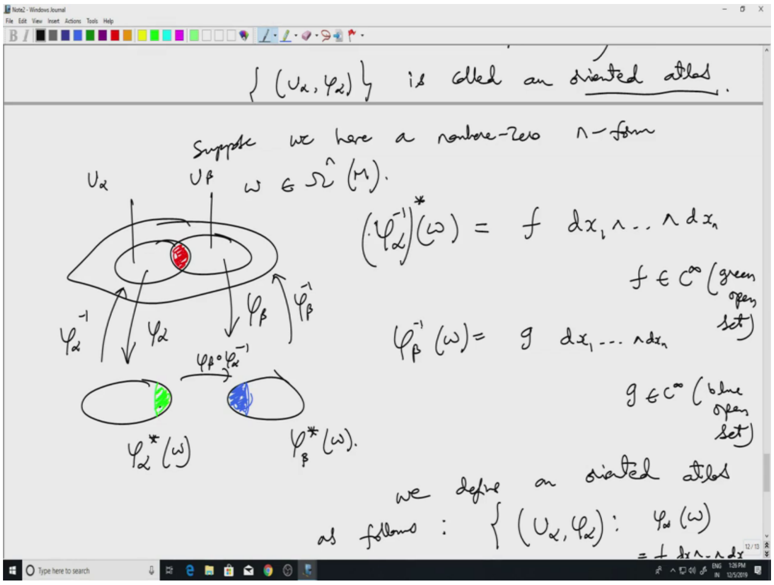

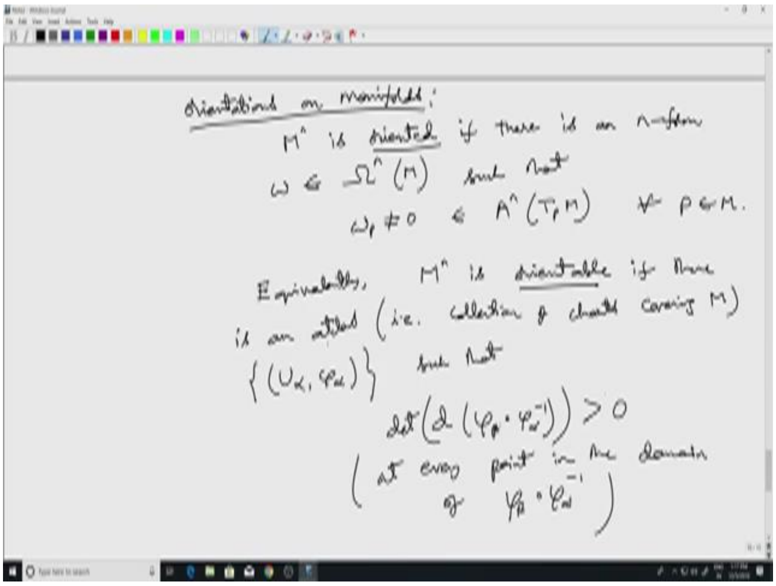

Now, here this does not directly give me a specific orientation, that is why I call it orientable. However, I can use this to get an orientation on M. And so that is why I say equivalently. And conversely if M is oriented I can get a system, an atlas of charts. So this is called, when we have such an atlas, I call it an oriented atlas. So there are 2 ways of thinking about orientation, one is to talk about n forms, the other one is in terms of transition functions, rather, the determinant of the derivatives of transition functions. And the 2 are equivalent. So let us quickly see why they are the same. Suppose we have a nowhere-zero n form omega n, the top degree n form, so I am starting with this.

Then let us see from this, how we can get an oriented Atlas. Well, let us draw our usual picture. So this is, one is U alpha, U beta or rather let me just instead of starting like this, I will say that, yeah, okay, let me continue with this and then I will give the definition later on. And here I have phi alpha, phi beta and as usual we are concerned with the red region, it is where the transition function is defined. So phi beta, so here I am doing phi beta compose with phi alpha inverse, we will take the green region to the red region, and I am looking at the derivative of that and the determinant of the derivative. Well, what I can do is, I can pull back the form, omega is on the whole manifold, so I can pull it back to get phi upper star Omega here and here I get the phi beta star omega.

Now, these are n forms on open subsets of Rn, different possibilities, different open subsets the green part and the blue part. At any rate, they are open subsets of Rn and these are n forms. So, therefore, I can write phi alpha star omega, at every point in the green portion it will be, we know that the space of n forms is 1 dimensional for a vector space. So if I take any point, this form will be a scalar multiple of the standard form dx1 wedge dxn. But as usual, the scalar multiple will change from point to point. So I can write it as a function times dx 1 dxn so this f is a function on C infinity of this green part, I will just write green open set and phi beta star omega be another function times dx 1, dxn, where g is C infinity of the blue open set.

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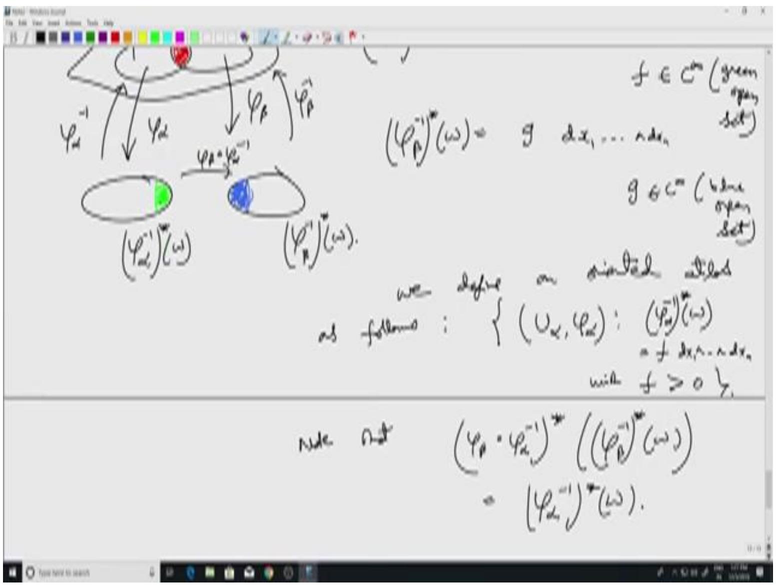

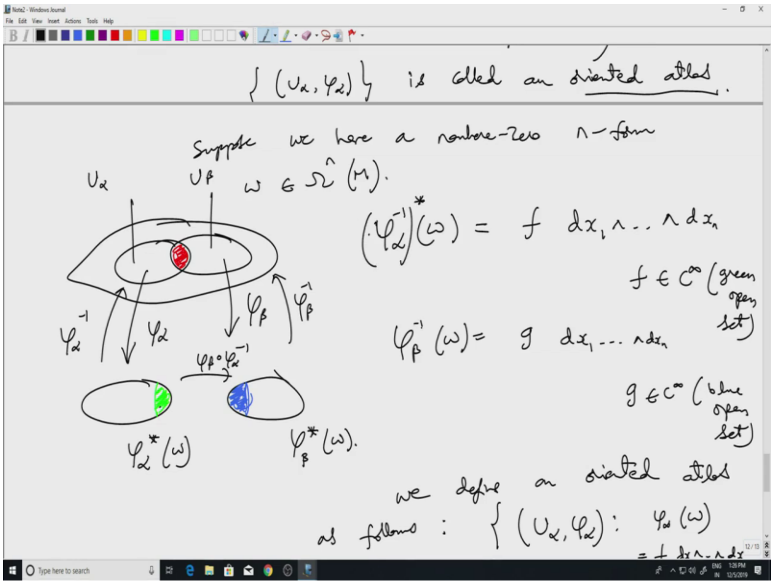

Now what I will do is I will just say that, I will look at suppose I define, we define an oriented atlas by, it is a collection of all charts, U alpha, phi alpha such that when I pull back this omega, I will get this thing here with f is strictly positive. So it is, a this function, in any case, it is a non-zero function because this omega is in non-zero form and this phi alpha star is an isomorphism of the alternating mappings. So this is a non-zero form, so therefore this f has to be non-zero. So I want it to be, I look for those for which it is just positive. Now, this gives, well this gives us a system of charts on M. It is not clear yet that I can even get an Atlas.

And the other thing is for this, why is it the case that the transition functions have the required property. So let us just check the transition functions. Transition functions condition. That is just a matter of relating this f and g. So notice that well, phi alpha inverse this map, the transition function map, I can use this upper star to pull back this form to this form I had to pull back this form here. And not surprisingly, the pullback of this will give me this because anyway, the whole thing was obtained just from omega. So it is not surprise that this is equal to, you can directly do the calculation here in fact, so phi alpha inverse and then phi beta, phi beta upper star, sorry, here, actually, I should be a bit careful.

It is not the I should use phi alpha inverse rather than phi beta, so inverse. So here everywhere, I should change it a bit. So phi alpha inverse star and here too phi alpha inverse star, phi alpha inverse star and here it would be star and here too, this. Because, yeah, I would need to go back to this and then phi alpha inverse star will give me something here and so on. Well if I do this calculation, this phi beta inverse star will cancel, the order will get reversed after I take star and then it is easy to check that, I will I just get phi alpha inverse star of omega. So in other words, this diffeomorphism between, the point is that, now this equation everything all the objects are defined on Euclidean space.

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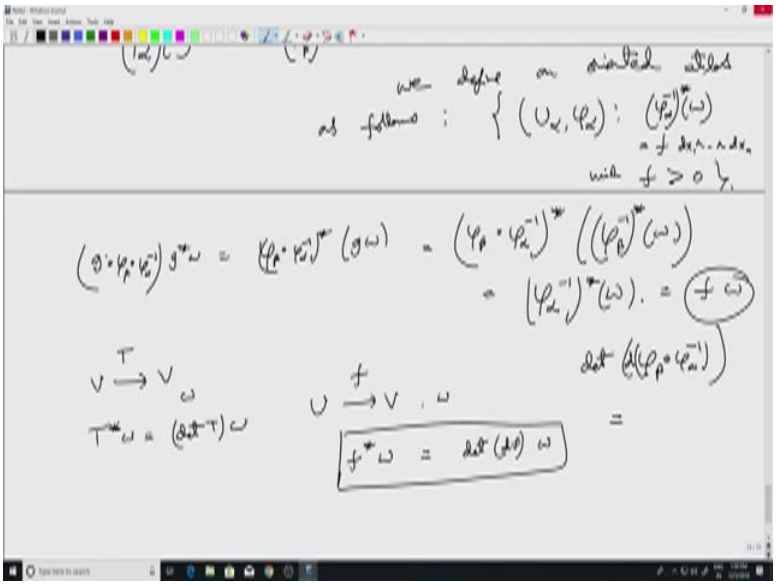

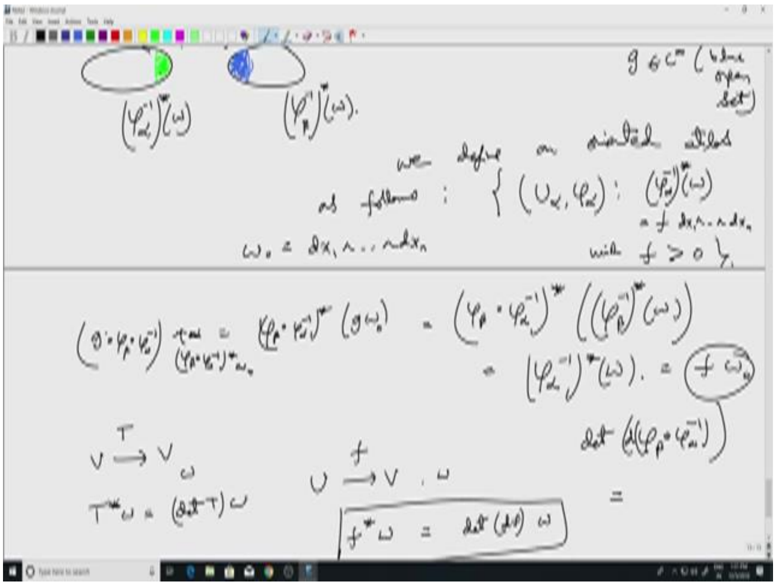

And, but we know that whenever we have a diffeomorphism between 2 open sets and if you have n form here, actually it is a vector space statement but it translates to this f star. So, here omega is here, f star omega and I do not even need diffeomorphisms, for this, I just need U and V to be open sets in Rn omega f star omega equal to determinant of df, well, times omega. This is Valid only for n forms. So we have something like this and this again can be reduced to a vector space statement.

We have proved that whenever you have a linear transformation from V to V, and omega is an n form, T star omega as determinant of T times omega. It is the same thing here, which I am using here. So, in short, the determinant and here the map is the f here is this map. Therefore, determinant of derivative of this transition function is well, now what we notice that this, the left hand side on the other hand is phi beta composed with phi alpha inverse star.

Now this thing is what we called g omega and this is what we called f omega. And this as usual when I do this, it becomes the pullback of this function which is g composed with phi beta composed with phi alpha inverse multiplied by g star omega. And the right hand side is f omega here and no sorry, not that. This it is not omega here, it is the standard form. Let me give it a name instead of writing it every time. So here omega naught as dx1, dxn. So I have just written this in terms of g omega naught. So here, let me just end with this calculation and then the pullback of. So, where was I? Yeah, fine. So this equal to this. So, this is times phi beta phi inverse star omega 0 is what I get here.

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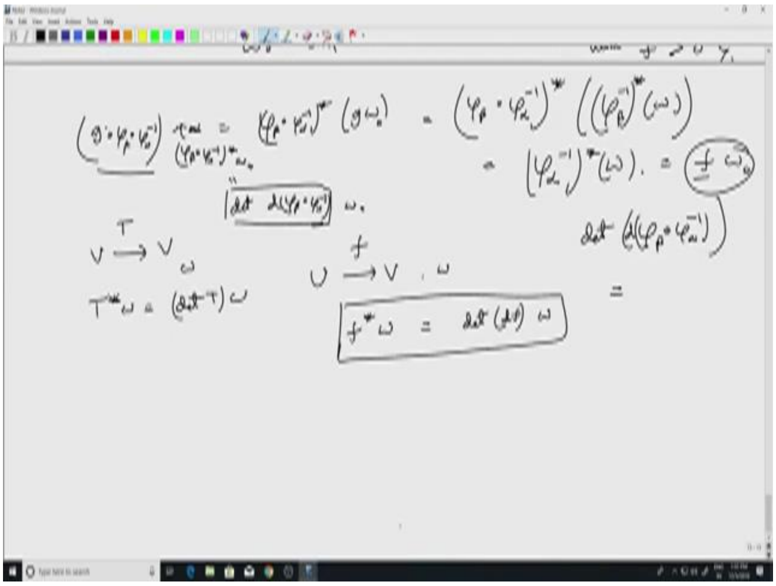

So, let me, so ultimately, and what we have to observe is that this again this thing is determinant of d of phi beta composed with phi alpha inverse times omega naught. So, the left hand side will be a product of this and this and the right hand side is just f. But since we have assumed both f and g are positive for my atlas, it will force this determinant also to be positive. So, therefore I get an oriented atlas. So I will stop here. In the last lecture, I will make a few more remarks about this. And then we will end there. Okay. Thank you.