An Introduction to Smooth Manifolds
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Lecture 42
Lie Algebras of Matrix Groups 20

Welcome to the 42nd lecture in this week, so last time we talked about the lie bracket term, the general linear root GL and R and saw that the lie bracket of left invariant vector fields corresponds to just the usual commutator of matrices ab minus ba.

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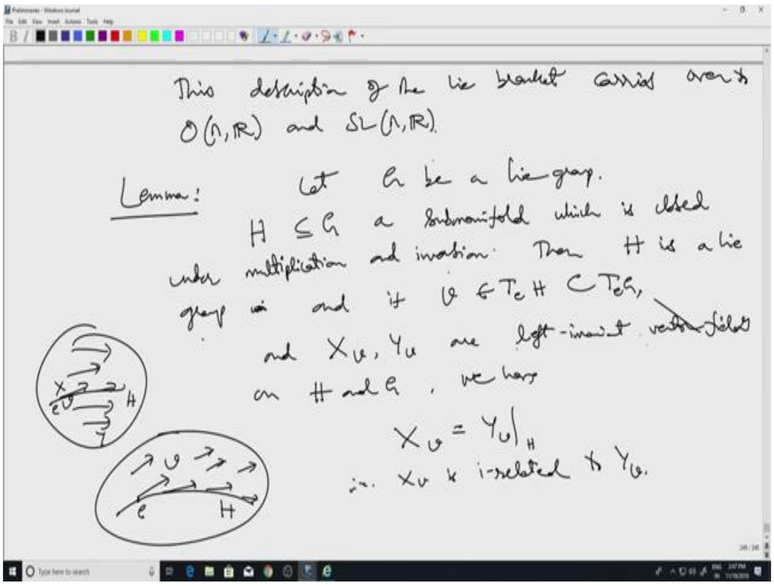

Now this is, I said that this description of the lie bracket overs to the subgroups over O n R and SL n R. So towards the end of the lecture I made this more general statement that let G be a lie group, H be a sub manifold which is closed under multiplication and inversion, then H is a lie group. So I should add that, so this is special, I mean this would be, it should be natural to call H a l subgroup of G, since it is a sub manifold and a subgroup.

But actually the way I defined a sub manifold is somewhat restricted and what is generally called a lie subgroup in the literature is something more general than this. But what I have defined is certainly special class of lie subgroups and so but for us since we are interested in describing the lie bracket on the subgroups, the important thing is the following that if I start with a tangent vector at identity of in the tangent vector to H, so v belongs to tangent the which is a subspace of TeG.

Now this v, so in the picture let us say this is the identity and this is the vector v. Let say this is H, this is e and this is v. I can extend this v, since v is a tangent vector in both, for both H and G, you can extent it to left invariant vector, left invariant vector fields on G in which case I will get a vector field on the whole manifold or I can extend it to left invariant vector field on H. So these two are what I call Xv and Yv on H and G. Then the claim is that this Xv the left invariant vector field on H is just the restriction of the left invariant vector field on G.

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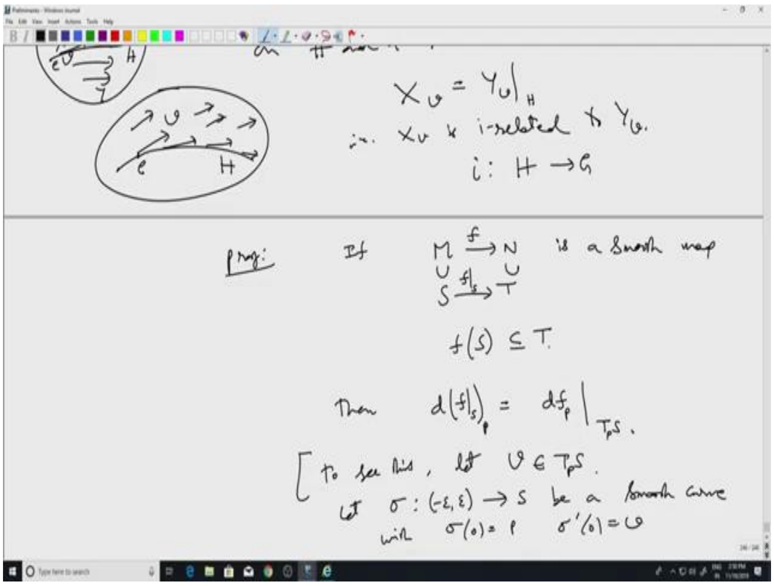

So and this can also be in the language of f related, this is i related, i is the inclusion map from H to G. And the proof is immediate because, well if you have a map between two smooth, manifolds, if M to N is a smooth map, suppose I have sub manifolds S here and T here. And S actually takes this sub manifold inside this, so in other words f of S contained in T. Then this if I look at the derivative of f at a point in S dfP, well now there are two f's, one is, so this f I should say f restricted to S.

So let me write it more clearly, so this is f restricted to S. So in fact I can talk about two derivative, maps of derivatives of two maps. Well one is df restricted to S at P and the other one is I can look at df at P and then restricted to the tangent space to S. The two will be the same, and this is immediate, one can see this, so all one has to do to see this is, see basically I can either use the slice description of the sub manifold or more invariant way of doing it is to just take a realization.

So I act it on, to see this you start with a tangent vector n to S. We know that every tangent vector can be realised as the derivative of a smooth curve. Let, sigma from minus epsilon, epsilon to S be a smooth curve with sigma 0 equals P, sigma prime 0 equals V.

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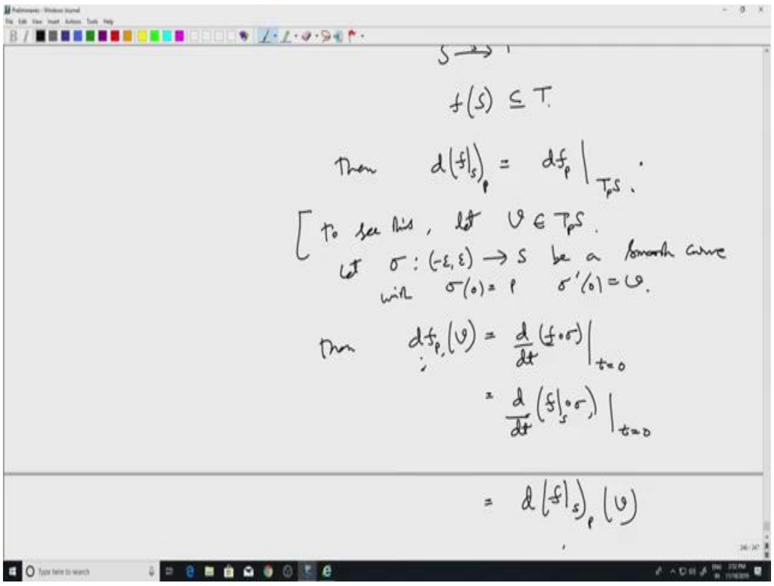

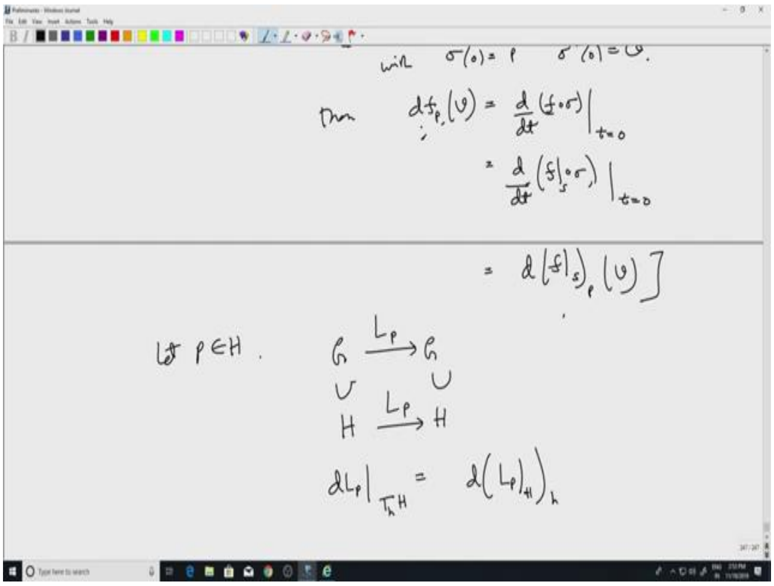

Then dfP acting on v is the same thing as d by dt of f composed with sigma at t equals 0. But notice that f compose, since sigma is curve lying in S, lying entirely in S, f composed with sigma is the same thing as I can write it as f restricted to S composed with sigma since sigma is entirely in S and this thing here again using the definition of, using the fact that the derivative, the action of derivative on a tangent vector can be interpreted in terms of what it does to a smooth curve, this is the same as d of f composed with S at the point P acting on V. So this equal to this, which is what we wanted to proof.

So here on the right hand side is this this thing here is the same as this and what I got on the right hand side, here is the left hand side. So that is the reason why that happens. So in short, when I have a sub manifold and I want to look at the restriction of a smooth map, I can either look at the restriction, take the derivative, or take the derivative and then just restrict it to the tangent space to the sub manifold.

So let us just do this to the, let us do this to the left translations. So G let us look at a left translation LP, so here I start with, let P in H, so left translation, now the point is since P is in H, left translation by P takes the subgroup back to itself. So from what we just discussed dLP restricted to the tangent space at any point ThH is the same as dLP restricted to H and then h.

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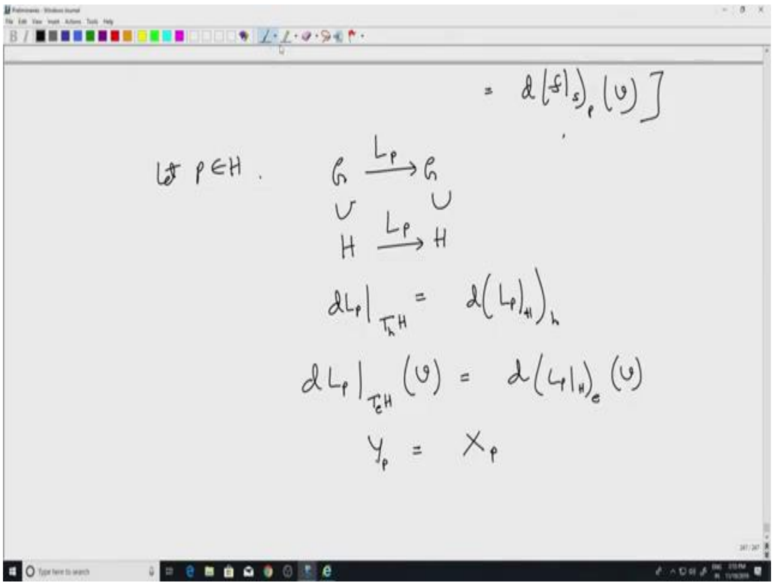

So if we act it on, if we act it on the vector v, so here I take H to be identity. So now let us take H to be identity, the equals dLP restricted to H of v. Now this thing here is just the restriction of Y, so this is YP and this is just YP, the left hand side and the right hand side is just XP. So I am just looking at left translation on H and then starting with the vector v pushing it all around the manifold by this derivative, so I get XP. So that proves what we wanted.

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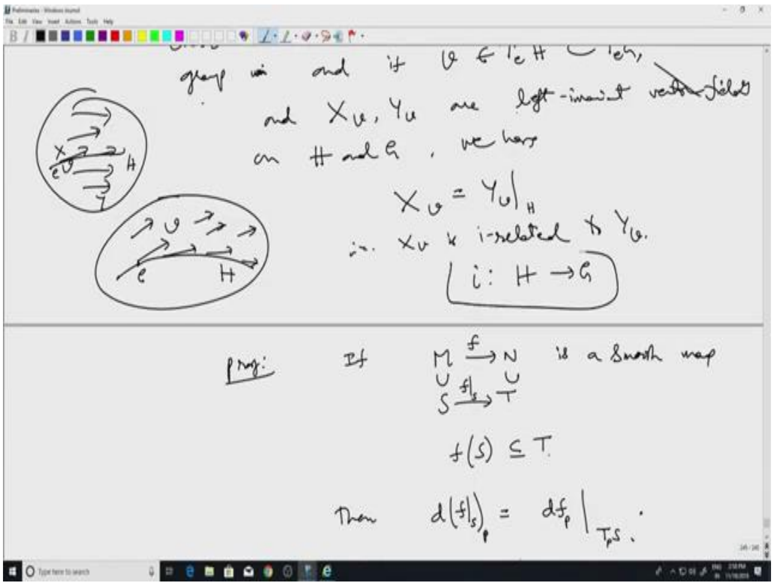

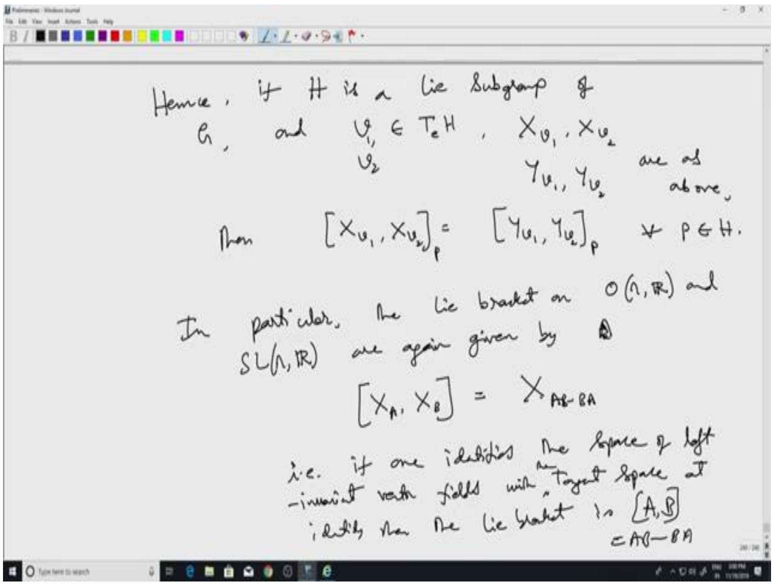

So once we have this, hence so this completes the proof of the small lemma. Hence, if H is a lie subgroup, again lie subgroup for us would just mean sub manifold which is also a subgroup of GL n, R. Well let me state the more general thing, if H is a lie subgroup of G, v belongs to the tangent space at identity of H. Let us say v1, v2 I will take two vectors, Xv1, Xv2, Yv1, Yv2 are as above, then Xv1, Xv2 is nothing but the restriction of this vector field to restriction of this vector field is, so in other words for all P in H.

So in short I can just forget about H, look at full left invariant vector field here generated by v1 and same thing for v2 which I called Yv1, Yv2, take the lie bracket. That lie bracket precisely gives the lie bracket of this smaller left invariant vector fields. And this again as a fact that this I related business. So applying, hence in particular the lie bracket on O n, R and

SL n, R are again given by, so if I start with XA XB, the lie bracket of XA XB is X AB minus BA.

Because the point is that if I look at YA YB in the notation we just talked about (YA minus), the lie bracket of YA YB is given by Y AB minus BA but X AB minus A is nothing but restriction of Y AB minus BA to the subgroup. So which is X AB minus BA, so in short if one i.e., if one identifies the lie algebra the space of left invariant vector fields with tangent spaces, with the tangent space at identity then the lie bracket is AB minus BA. So exactly whatever happens for GL n, R happens for this. Well that concludes our discussion so this example is one example which illustrates everything very clearly.

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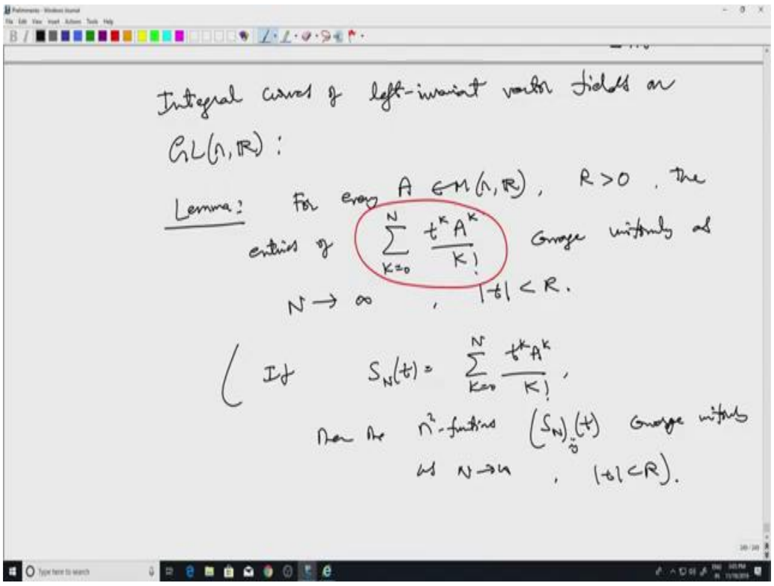

Now the other thing I wanted to say before I move on to some general stuff about lie bracket. Let me in, the again staying within the realm of matrix groups in particular gl n, R. We can also describe, what left invariant vector fields look like, we have seen what lie bracket looks like, we can also describe what integral curves look like in GL n, R. So let us do the integral curves for left invariant vector fields.

Integral curves of left invariant vector fields on GL n, R So for this I need a Lemma what exponential of matrices, so lemma, for every A in M n, R and any R greater than 0, the entries of, so I am, entries of this matrix t to the power k A to the power k by k factorial, k equal to 0 to let say capital N converge uniformly as N goes to infinity on the set and mod t less than R.

So this, here capital N, for every capital N I get a finite sum of these matrices t to the power k, A to the power k or k factorial. Now, so this will give me an again an element of M n, R, so I look at the N squared entries of this matrix, of this matrix, each entry is a, so I get sequence of functions, actually I get N squared functions.

So if I call this, this partial sum has SN is equal to, SNt is equal to k equal to 0 to N t to the power k by k factorial, then this N squared functions SN the ijth entry of this t converge uniformly as N goes to infinity and mod t less than R in this open interval R can be as large as we want, whatever R is on this open interval these things converge uniformly. And therefore one can and notice that this SN ijt just power series in t. So partial sums of power series for t. In other words, it will just involve powers of t.

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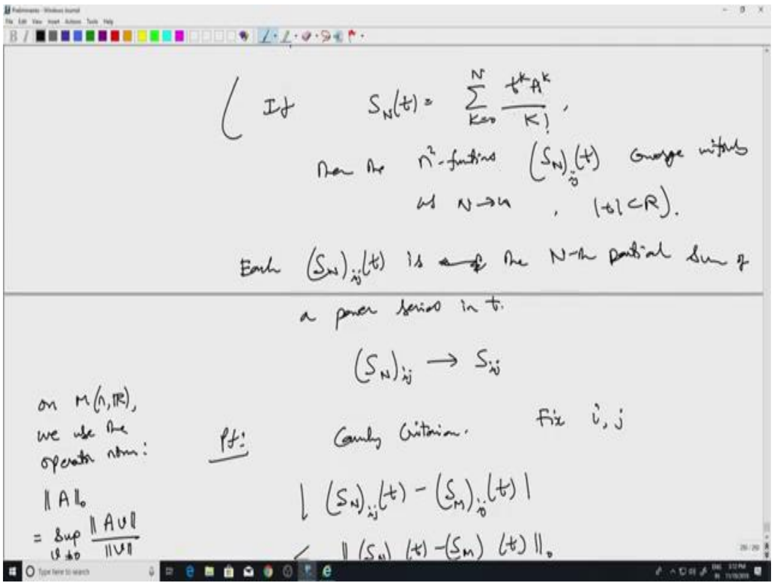

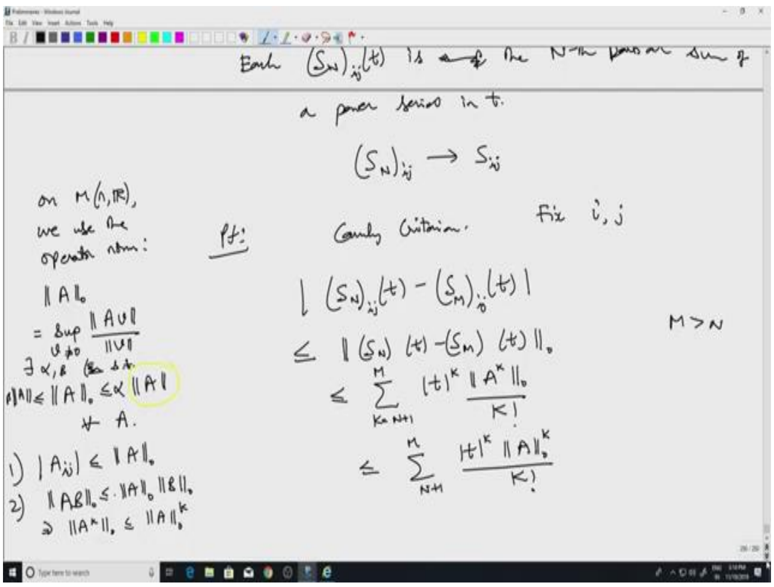

So each SN ijt is a, is the nth partial sum of a power series in t. The good thing about the power series is that if once we know that if we have uniform conversions, the limit will give me a function, so again which will depend on ij. So the limit give me a function which I will call Sij. So we know that SN ij this will converge to some Sij, again it is a function in t.

And since we have uniform convergence this will be differentiable, Sij will be differentiable infinitely many times and in fact to obtain the derivatives of Sij I can just do term by term differentiation here and take the limit. So I will get this. So these are general facts about power series of one variable.

Now, so now let me quickly proof the Lemma. We just use the quasi criteria, in other words we estimate, let us do SN ij t minus SM ij t, let us estimate the size of this. We want to say that when capital N and capital M are very large this can be made small and that very large should not depend on t, it can depend on this R but not the specific value of t.

So now let us, this will be less than, so what I am going to do is, I will use on M n, R we use the operator norm. So the norm of a matrix I will define to be sup Av and then sup Av so here I can put v, v nought equal to 0. So this turns out to be, this is equivalent to the Euclidian norm, so in other words, so let us put O here, so this is this A norm is less than or equal to, there exist alpha beta such that for all A, whatever A we take, the same alpha beta will work for all A.

So I should, there is no b there, it is just a, A here again. So for all A the same alpha beta will work, so I can basically there two norms are comparable which are matter of convenience. So and it turns out that the ijth entry of a matrix, so mod Aij the ijth entry of a matrix will be less than or equal to the full norm of operator norm from matrix. The Euclidian norm if the matrix here, the right hand side what I have written here this part is just the square root of the, you take the square all the entries of A, add them up and take the square root, that is the Euclidian norm.

Now this one is, it is a fact that Aij is less than or equal to this and one can see this just by acting A on certain appropriate vectors. Well, so let us use this and so write it as SN ij t minus SM ij t, so here I should say fix i and j on this the operator norm. And so which is less than or equal to, since it is a norm triangle inequality satisfied and therefore I have, actually sorry so here there is no ij, I am looking at the full matrix.

So now the partial sum SNt is given by this, so I will use triangle inequality and take the sum norm inside the sum si, so I will get less than or equal to submission k equals, well assume M is bigger than N, N to N plus 1 to M and then modulus of t to the power k operator norm of A 0 to the power k, operator norm of A 0 to the power k. And then I still have this k factor here, now here is where is this operator norm comes into play. So this operator norm has the nice property that the norm of AB is less than or equal to norm A times norm B.

That follows immediately from the definition and while the Euclidian norm does not quite have this property, constant will appear before this on the right hand side which is not, I mean which is not a big problem for our proposal but this simplifies calculations. So this just turns out to be norm AB less than or equals to norm A times norm b. So if I keep using this repeatedly, then I will get norm of Ak less than or equal to norm A and the kth power of that. So this is less than or equal to N plus 1 to M modulus of t to the power k, and then I can take norm of A and then raise it to power k divided by k factorial.

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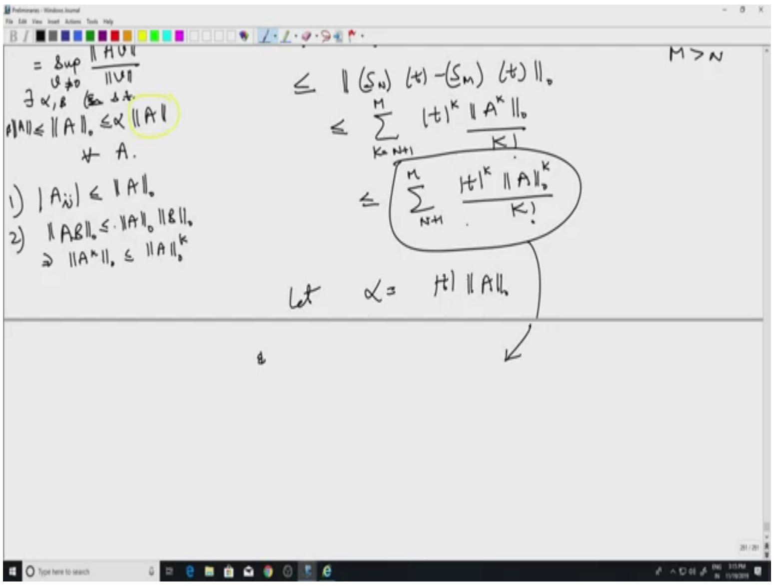

Well, let us call this, let alpha equals mod t times norm A naught. This is what I have written here, this thing is just the difference in partial sums for a new series of a, this time for, yeah actually perhaps I will stop here and just complete this the next time. Thank you.