An Introduction to Smooth Manifolds
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Lie Brackets (Part 1 of 2)

So hello and welcome to our continuing ongoing lectures on Vector Fields.

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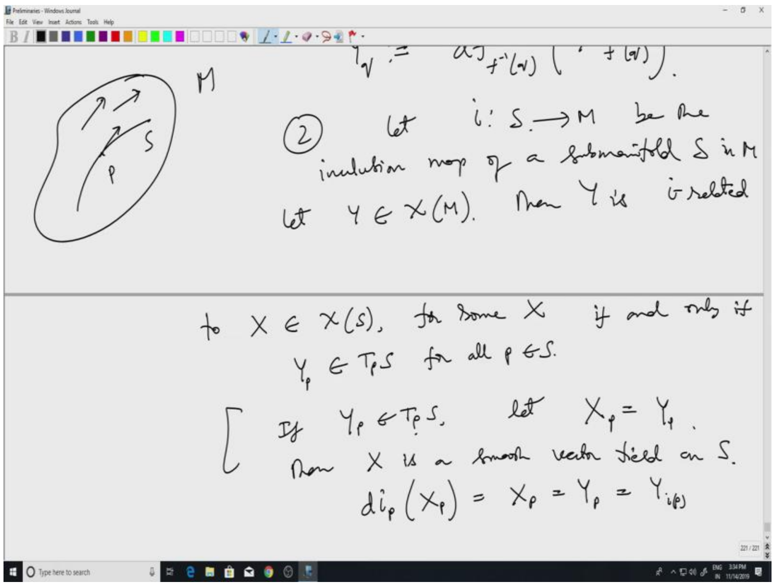

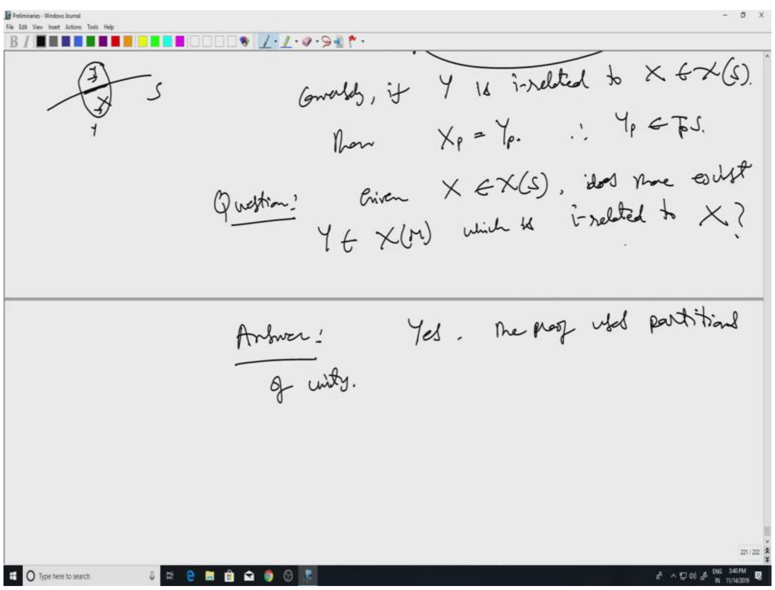

So now, last time, I had introduced this notion of f related vector fields, where f is a smooth map. Now, let us over here looking at the inclusion map of a sub manifold and a manifold M. So, the first thing is, let Y be any vector field on the big manifold. Now, what does it mean to say that Y is f, i related to something on S, it just means that as it turns out, then Y is i related to X. Oops, perhaps I should start the next page.

Then Y is i related to X for some X, if and only if, if and only if Y at the point p, so here is the big manifold M and the sub manifold S, so Y is bunch of arrows, so p is a point on S. So I would like this Y to be tangent to S at p, if and only if Yp belongs to TpS for all p and S. The, so if this an if and only if statement.

So one way is that suppose, this condition holds that Yp is tangent to TpS for all p in S, then we can define, if Yp belongs to TpS, let, you can define a vector field let Xp equal to just Yp. Then it is from what we have said already regarding smoothness of maps and so on, or restricted to sub manifolds, then this construction, just restricting it to a sub manifold, then X is a smooth vector field on S.

The point is that, this, to say that something is a smooth vector field on S, I should need, I need the condition that whatever I am going to define should be in the tangent space of S, not the tangent space of M. And I am assuming that condition already here, Yp belongs to Tps, I am just calling it a different name (expl) so I could have very well said that let X equal to Y restricted to S that is fine as well.

So I get a vector field and now, this is almost trivially i related to, so because what I want to check is dip Xp. Now, the derivative of the inclusion map is just the inclusion map on the level of tangent spaces, so in other words, it is like identity except that it is on the same space, it is a sub space. So dip of Xp is just Xp itself, but Xp is the same as Yp and this is Y, I can write as Y at ip.

So the way I have written it, it is, I have set it up so the i related condition is, holds. So, this equal to that is the i related condition. And, this equation, well, it also tells us that conversely, suppose I have, right, what would be the converse, the (con) it would be that, if Y is related to, i related to some X, then Y is tangent. So, suppose Y is i related to converse, if Y is related to X which is tangent to the sub manifold.

Then, again look at this condition, that i related condition that I have written here, what does it tell us, well this, the left most side has to be equal to the right most side but again, the left most side is equal to this because the derivative of inclusion is the inclusion at the level of tangent space so this is Xp.

So everything whatever I have written here continues to hold. And that is Xp and that should be equal to Yip which is the same as Yp. It is just that the order in which I write the equalities will get switched around a bit. So I will end up, but I will end up getting the same thing, d rather that, Xp would actually be equal to Yp. So in particular, therefore, Yp would (bel) be tangent to this. These are all trivial statements that I am making here just going through the definitions.

But what would be more interesting is, if I, another question one can ask is, so question: so given X and S, does there exist Y in the big manifold which is i related to X. So, as we have seen i related to X just amounts to saying that the restriction of Y to the sub manifold is X. Now, so what we are asking is, given a vector field on the sub manifold can we extend it to a vector field on the big manifold.

And it turns out the answer is yes, but we one, we need an ingredient which I have not discussed yet, so I will not prove it right now but I will just so state it. Answer: Proof uses what are called partitions of unity which are sort of, basically you, we have seen the use this cut off functions, infinity functions with compact support, we have used it often.

This is something, one starts with such things and then builds up some other object which enables us to the main use of such as partition of unity is, whenever things are defined locally, you want to define it on the whole manifold, you sort of patch up the local things and get it.

Here, for instance, locally it is always possible to extend a vector field, so if I have a sub manifold, and let us say this X is here, I can always extend X to a locally to something Y and the reason I can do this is I just use a slice chart and in the slice chart I sort of make a define Y to be such that it is sort of constant along the, you, it is basically X, Y should be X along all the horizontal slices. Roughly that is the idea but, we will not be needing this fact, this extension, so I will not prove this.

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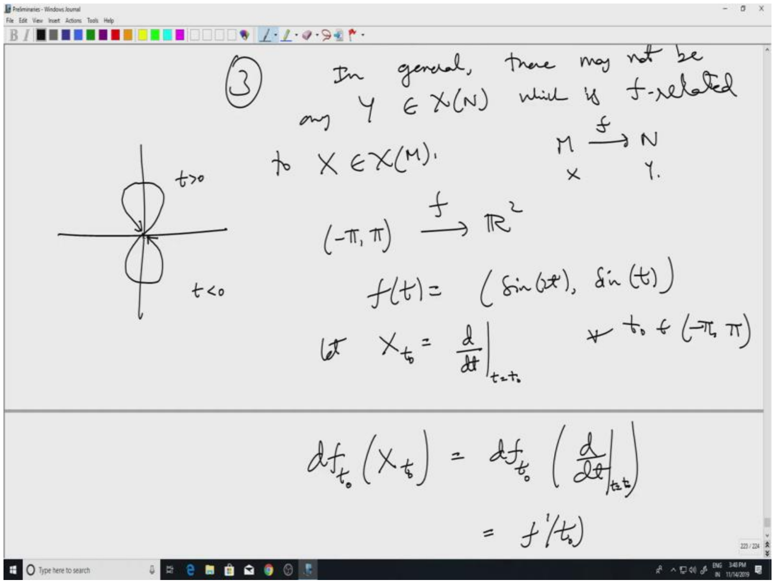

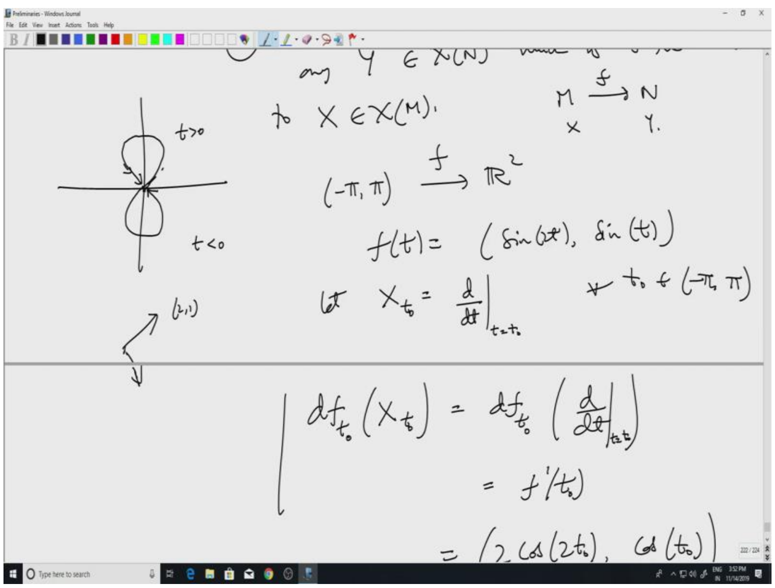

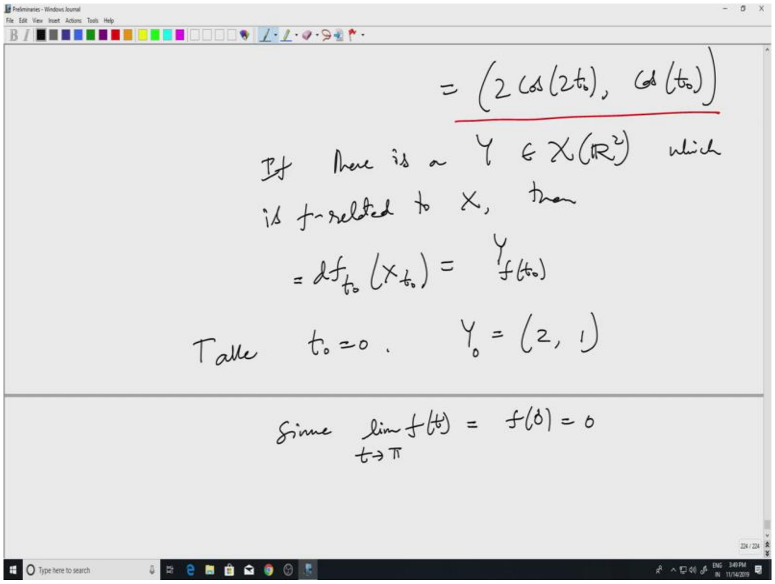

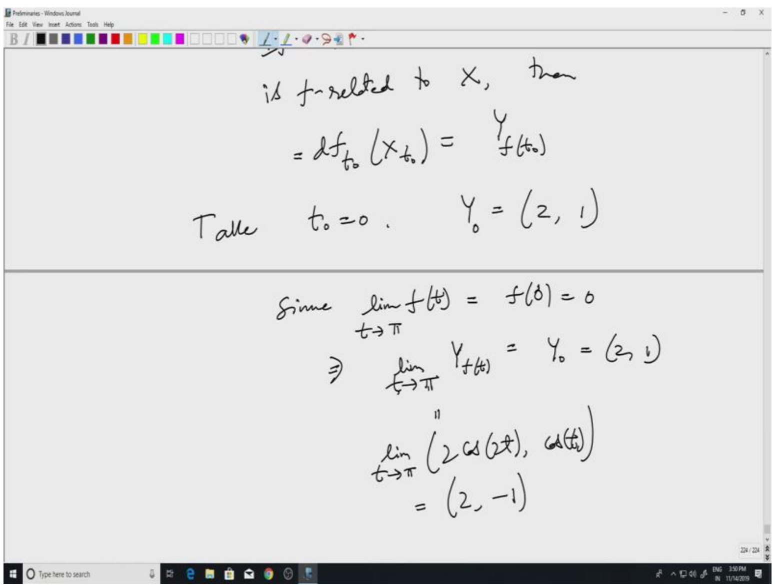

Now, another important sort of example is the following. If in general, let us see an extreme case, there may not be, there may not be any Y in the target manifold which is f related to X and M. So the setup is the same, so f from M to N, there is an X here, and we are looking for a Y. So, given X, there may not be anything in, there may not be a Y in the target which is f related to X. And here is an example, so let us look at the map, from minus pi to pi the interval to R2 if f of t is sin 2t and then sin t. So this is a curve which looks like figure eight.

So, t0, it is 0, it starts at the origin. As t increases, it will stay in the, it will go like this. Well, my drawing skills are not that great, so I have to do it again so, it will go like this. Let me put an arrow here to indicate that, so here this is as t goes to pi, I will move along this arrow. Now, as t, this is t equal 0 so as t becomes negative it starts going down and again it comes back to this. So this corresponds to t greater than 0, this is t less than 0.

So let, so I want to say that this, let us, so my domain manifold is just an interval. Target manifold is R2, so on the domain manifold I let us take the vector field, let Xt equal to, I just take the, on any open interval there is a natural vector field namely the standard derivative operator d by dt, d by dt at the point, so let us call it t naught, at t equals t naught for all t naught in minus pi to pi I take this, which is actually just the standard basis for the tangent space at each point.

Now, what does it mean to say that something is f related? So I need to just, let us look at dft of Xt equals dft of d by dt. This is what we called this was our definition of f prime t. So maybe I should put t naught just to be clear, so let me put t equals t naught here, so this will be t naught. So, and we know that, this in the case of us regarding, I mean, if you regard f as a curve in R2, this notion of derivative coincides with our usual notion of derivative where we just differentiate this component wise this curve.

So that will give us 2 cos 2t naught, sin of, oops, no, that is cos of t naught. This looks a bit hard to read so let me just rewrite this whole thing again. So this is, 2 cos 2t naught cos t naught. And this is supposed to be, if Y if, there exist, if there is a Y in a vector field on R2 which is f related to X then, this df t naught of X at t naught should give me Y at f of t naught. Then df t naught at X t naught should by Y by definition of f related.

So this is f of t naught and this is actually, so we have seen, but we have seen that the left-hand side is actually equal to this thing here. So I will not write it again, so the left-hand side is this, 2cos t naught and then cos t naught. But notice one thing that, so actually, so let us now take t naught to be so this all this stuff was done for any t naught. Now let us just take t naught to be 0, so in which case take t naught equal to 0 and then, what one would get is Y of 0, f of t naught would also be 0.

So, and here I will get, by plugging it in this formula here I will get, so cos 2pi, so this is 2,1, Y naught would be this vector. On the other hand, the same point, f of t naught is this origin is also the limit of f of t since f of t, limit of f of t as t goes to pi is also equal to f of t naught, f of 0 equal to 0 comma 0. Well, instead of writing 0 comma 0 I will just write a single 0, the origin. So since this holds, by continuity of Y, after all Y is is supposed to be a smooth vector field on R2, so it certainly is continuous.

This would imply that limit t going to pi of, limit t going to pi of Y of f of t would be equal to Y at 0 which we have seen is 2 comma 1. But, now arises the problem, so we already know that what, why at f of t is given by dft or it takes t naught and this is basically a cos 2, so the left-hand side is limit t going to pi 2cos 2t naught, 2t, cos t. As t goes to pi, this goes to a, the first term is okay but the second one is minus 1, so this would be 2 comma minus1 but it should also equal to 2 comma 1 which is impossible.

Now, geometrically it is clear enough after all as I, this equation shows that dft Xt is the same thing as the tangent vector to this curve at this point. So, since the curve is going like this, the tangent vector should point like this, this the vector 2 comma 1. On the other hand, the same point is being approached by the curve again from this side, so the tangent vectors to these, this portion of the curve will point in this direction.

But the tangent vector to this curve is precisely, the f related condition tells us that the tangent vector to this curve is precisely Y at that point if Y existed. So, at this problematic point, the origin, Y would have to assume at least 2 different values, in fact, if we go along to minus pi, we will get another one. So, Y cannot be, so, Y does not exist in short, which is f related to X. All right. So that proves, that in general it may not be possible to find f related vector fields.

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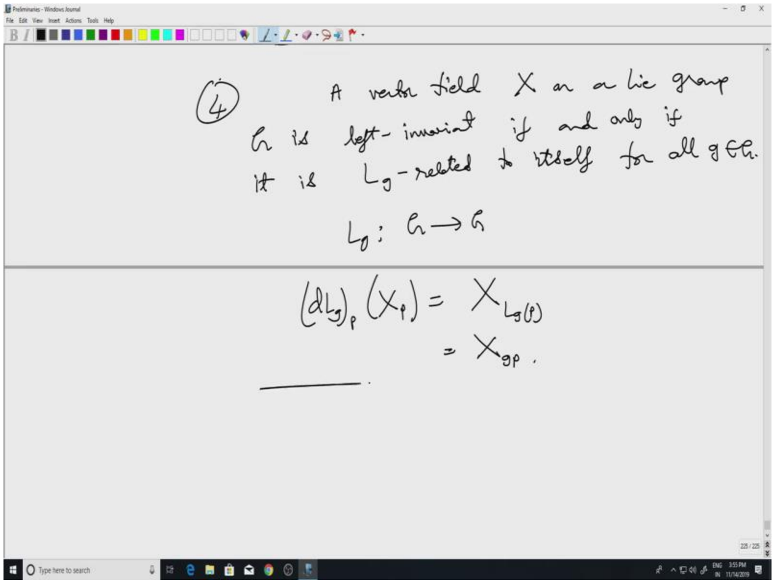

Now, I want to talk a bit about, now that I have introduced this notion of f related let me just mention one more thing, it is just a restatement of something that we already know. Namely, a vector field on a lie group, X on a lie group G is left invariant if and only if, only if it is Lg related to itself for all gG. So, this is just a restatement of something that we have already seen. This is in fact the definition of a left invariant vector field.

So because, what does this mean actually? So Lg, as we know, is a map from G to G left translation and it is a diffeomorphism in fact, but we do not need that. To say that, X is Lg related to itself just means that, dLg at any point p, X at p should be equal to X at Lg at p. So, since I said it is related to itself, X is making an appearance on the right side as well. Normally it would be Y but, and then, Lg of p is just, by definition it is just gp.

So in short, the Lg related condition is precisely the left invariance condition. A slight, I mean, we have defined Lg left invariant by using the identity, but we saw at, seen way back that, can use any point. So, Lg related is the same thing as this, left invariance and conversely left invariance gives us this property that this equation and that is the same as Lg related. So, just a restatement of what we have seen before.

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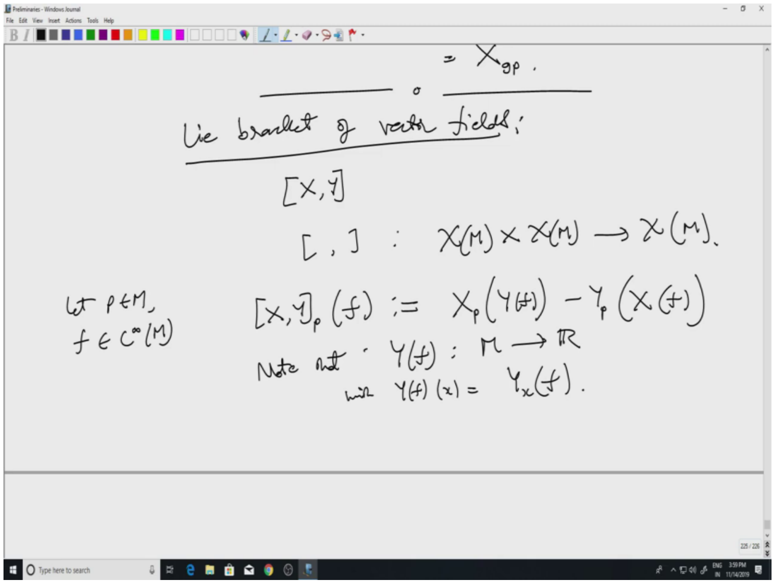

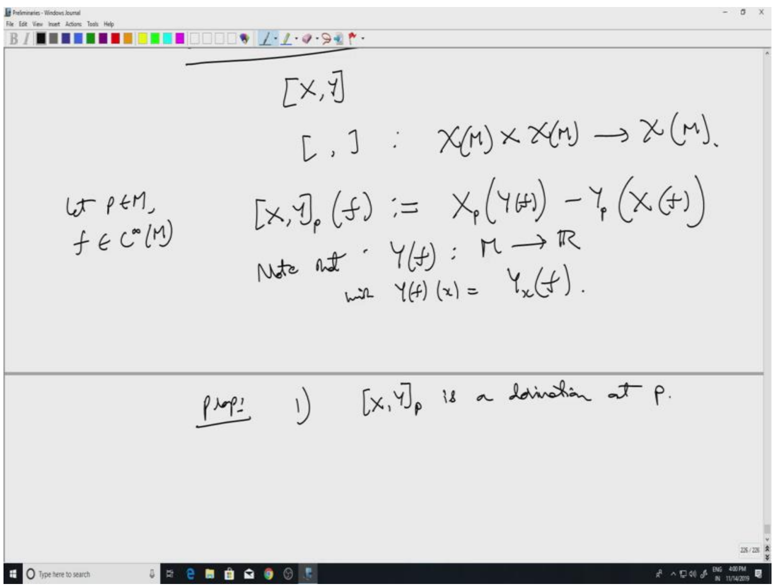

Now, before moving on to, moving on from the topic of vector fields, there is a fundamental construction operation involving vector fields which I would like to discuss for a bit. So, this is called a Lie bracket of vector fields. So this is an operation where the input is two vector fields, and using two vector fields we are going to get one more vector field out of that. So, this is denoted by X square brackets Y.

So this is a map, so this map is a function from vector fields on M cross vector fields on M back to vector fields on M. It is defined as follows, so, I want to say what the lie bracket of XY. So this is supposed to be another vector field, so let us take a point. Let p belonging, let p belong to M and this thing if I specify what it is at p, then I am done. So, to say what it is at p, I have to regard it as a derivation at p, so let us see what its action is on a C infinity function.

So this value at p acting on f, we define it to be this the definition. Well it is Xp Yf minus Yp Xf. Now, here what we have to note is that, note that, Yf, when I write this Yf, this is actually, so f is a C infinity function on M, Y is a vector field on M, the combination is (act) can be regarded as a function on M which I have, in fact we have used this function to talk about smoothness of Y. So this function, this is a function on M, its value at a point X is YX at f.

This is something that we had seen earlier and same thing with Xf as well. So, these two are functions then and again one is doing the, so it is like you are taking mixed partial derivatives. So, if we think of Y acting on f as taking directional derivative, then X acting on that is taking one more derivative. So it is like a mix directional derivative and this also but somehow the fact that we have changed the order of differentiation will end up giving us something non-trivial, in general.

So, in fact I have few things to say about when this will be 0 and so on. But in general this will not be 0 and the other thing I wanted to say was that. So this is what, this how it is defined, but there are things to check, one is that, first and foremost even before talking about smoothness one has to check that this thing here on the right side is in fact a derivation, is a derivation at p.

Once I know that it is a derivation, so I get a well, so I have a tangent vector, so in other words, this gives me a tangent vector. As of now it is just something starting with a smooth function, I have obtained a real number, but actually what I want to claim is that this acts like a derivation, so I have to check Leibniz's rule and linearity. So, I will do that and then say a few words about smoothness in my next lecture. So thanks.