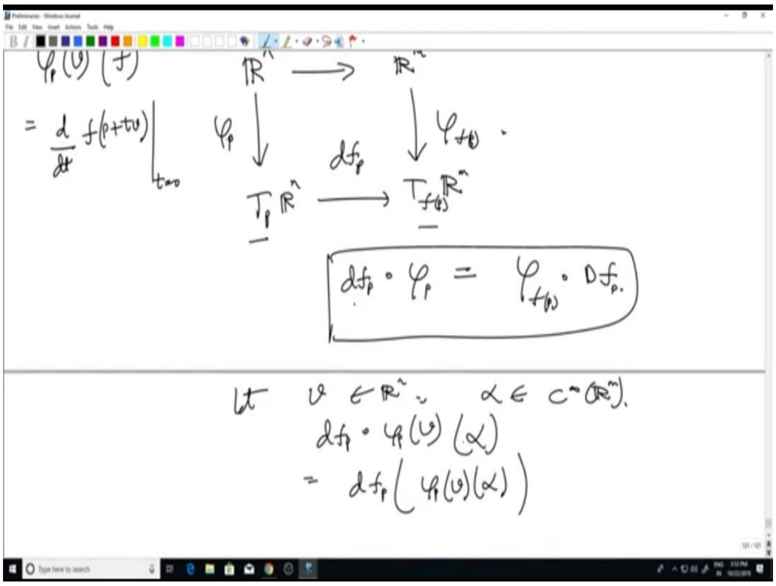
An Introduction to Smooth Manifolds
Professor Harish Seshadri
Department of Mathematics
Indian Institute of Science, Bengaluru
Lecture 20
Basis of tangent space

Welcome to the 20th lecture in this series. So, I was in the (())(0:34) a small computation last time. I wanted to show that the notion of differential of a smooth map between manifolds coincides with our earlier notion derivative when interpreted suitably.

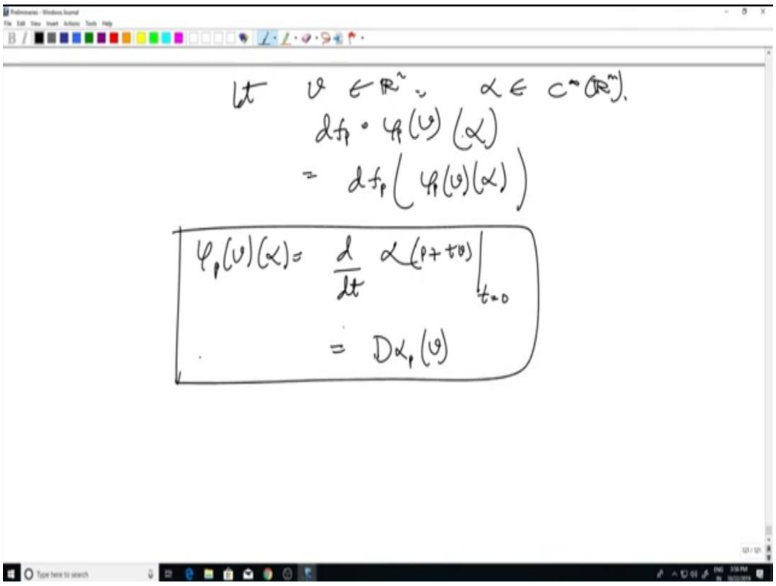
(Refer Slide Time: 00:52)

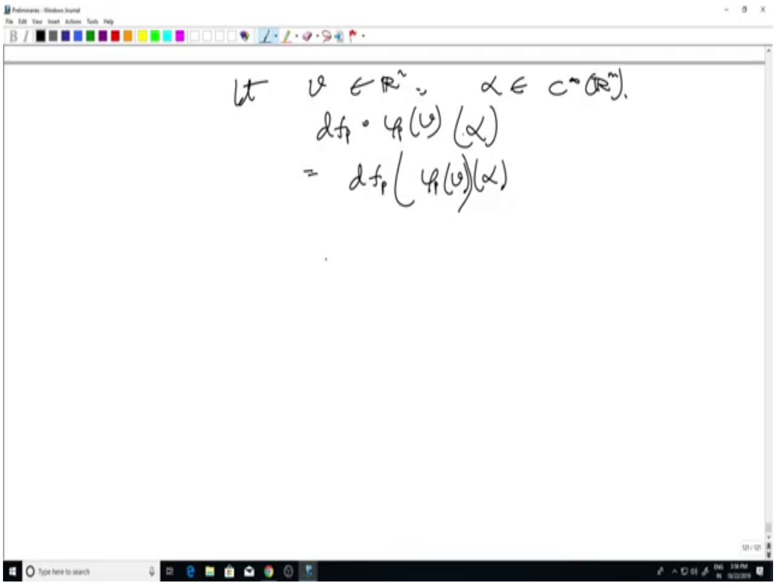


So, in other words, let us start with a smooth map between r n and r m. We already have the classical derivative and then we have this new derivative d f p and the claim is that when we identify r n with t p r n via this natural isomorphisms then this equation holds. To check this as I said if it start with v in r n, phi p v will land up here and then d f p will land here.

So, in other words what I get this thing here is actually derivation on r m. So, I have to see its action on a c infinity function on r m. So, I took alpha. So, let it act on alpha and that so let us unravel this. D f p acting on phi p v alpha, this is what I have here. Well phi p v acting on alpha as I have mentioned earlier.

(Refer Slide Time: 02:31)

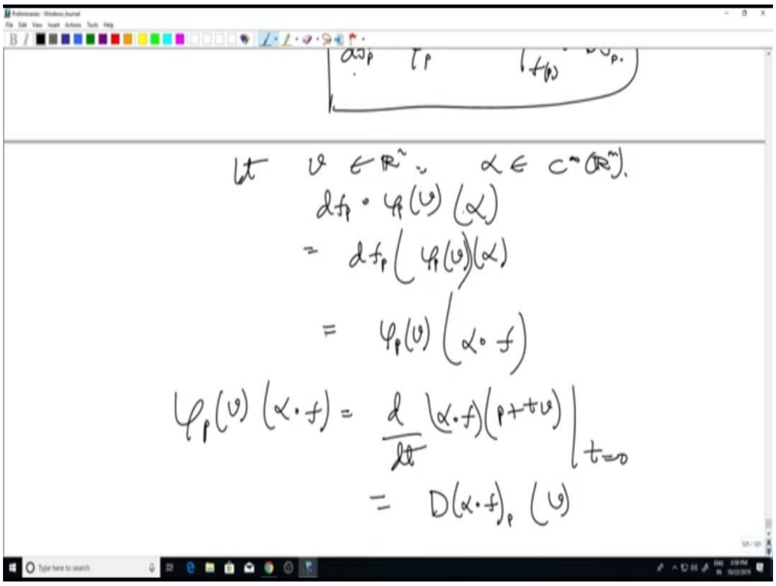




So, phi p v alpha is the directional derivative d by d t alpha of p plus t v. I do not need this bracket and t equals 0. That is what that is the definition and we also know that this is the same thing as d, the usual derivative. Right, so where was I? Yes. Yeah so this is one could write this in a couple of different ways. This is the directional derivative of alpha in the direction this and that is the same thing as now using my new notation. My new notation for the classical derivative I use capital D. So, capital D of alpha at the point p acting on v is what I have.

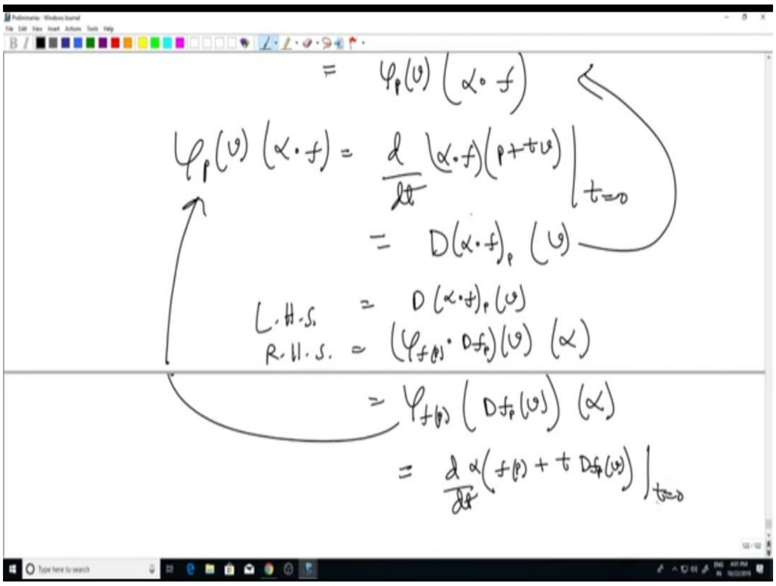
This is one thing but this is something to we will use later on but let us go back to this what I have here. We know that the differential of a smooth map in the abstract sense is given by so this is going to be, I think I kind of did not write it correctly here. It is not quite this d. The bracket should not come here. The bracket should come here. So, really I do not, I mean I use this but on the not at this stage, let me erase that I do not need it at this stage.

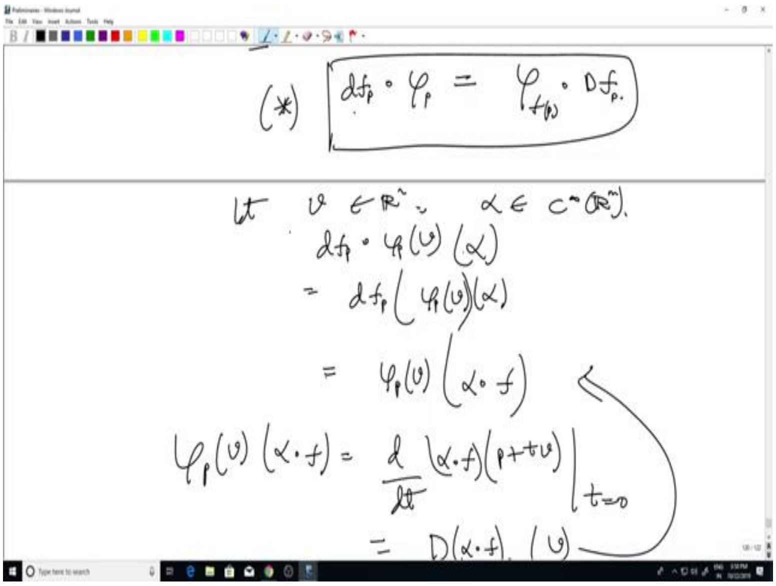
(Refer Slide Time: 05:06)



So, this is what I have and I know that this is the same thing as phi p v of alpha composed with f. What is this? This now I am going to use the definition of directional derivative. So, this is just the directional derivative of this function along the direction v. So, I know that this is phi p v acting on alpha composed with f is just the directional derivative d by d t alpha composed with f acting p plus t v at t equals 0 and I also know that this is the same as the classical derivative of alpha composed with f at the point p and then acting on v.

(Refer Slide Time: 06:18)

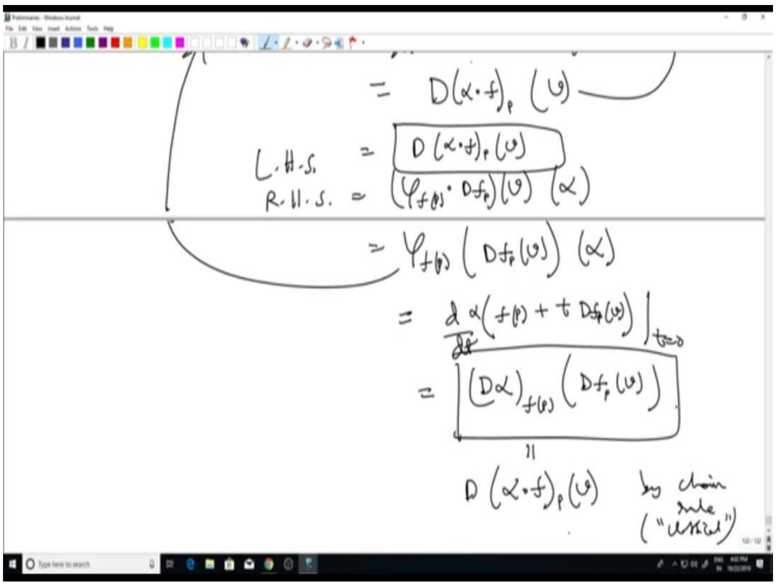




So, the in the end what I get is now I will just plug it back in here. So, what I get is the left hand side of this equation, let us call it star. L h s when acted, of course I have to act it on v and alpha. Then what I get is this exactly this D alpha composed with f at p of v. What about the right hand side? Right hand side is phi at f of p and then I have D f p. Now, the whole thing should act on v then eventually it should act on alpha.

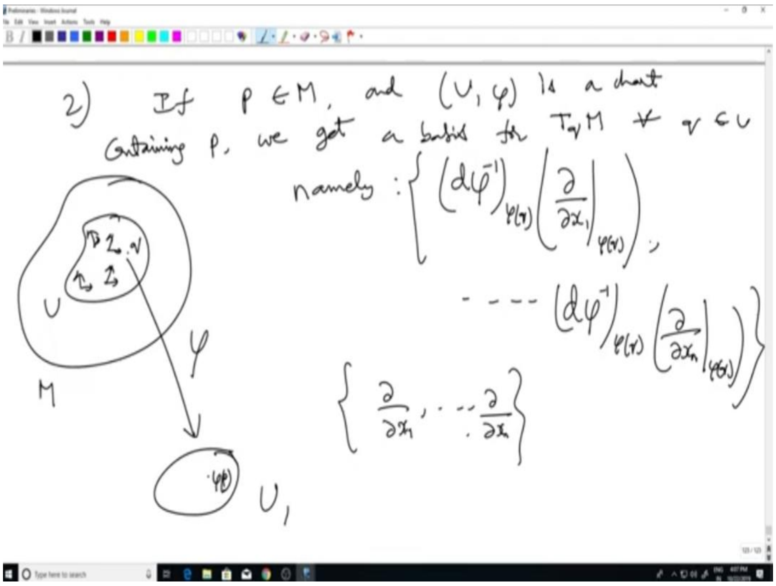
This is phi of f of p D f p acting on v acting on alpha and what does this. This is right and again using the same logic, so if I want to see what just like I had this here. This is phi at the subscript p f of p something something is equal to this is the same thing as d by d t of so essentially I have to just take. So, alpha the starting point is f of p and the vector is t times D f of p v at t equals 0 and what does that give us. Let us see. This is the same thing as right.

(Refer Slide Time: 09:15)



So, this is alpha, D alpha at the point f of p acting on a vector D f of p of v. Well, if you notice this expression here and this expression here they are the same by chain rule. So, this which is equal to and this chain rule that I am talking about is the classical chain rule. Chain rule classical. So, that using the classical chain rule, we see that everything works out and the new derivative new differential and the old one are related in the expected way. Now, let me so far it is been somewhat abstract.

(Refer Slide Time: 11:15)



Let me illustrate all this with some examples. So, first thing is yeah now I said, yeah before I move on to examples I should add one more very important thing. The first thing is the, I said that the proof shows that something like this happens. The other thing is that the second point is that if p belongs to m and u phi is a chart. This is something very important for computations. Chart containing p. We get a basis for t q m for all q in u. This is my m, this is u.

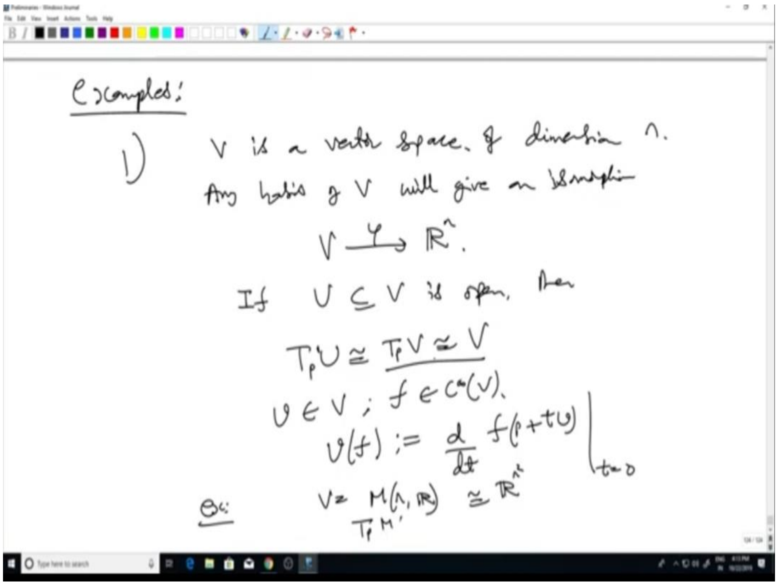
So, pictorially sort of at every point I have a basis for the tangent space and that basis is essentially so we saw that. This chart map phi is a diffeomorphism. Therefore, you set up a correspondence between tangent spaces here isomorphism between tangent space here and tangent space here.

So, you get a namely, so if I want to get a basis for t q m, I know that the derivative d phi is going to map it to the tangent space to this. So, this is u 1 which were identified by tangent space to r n itself and we know what natural basis for tangent space to r n is. Namely the derivations corresponding to partial derivatives along the e 1 e 2 e n directions. So, namely I look at d phi so this is the point q and this is phi of q. D phi inverse at phi of q.

Then I look at this derivation d by d x 1 at phi f q. This is what one single vector and so I will get n vectors like this. D phi inverse, phi of q d by d x n phi of q. So, this we have a natural basis for the tangent space of any open subset of Euclidean space namely the partial derivative operators, derivations and using these derivations I can sort of pull them back via d phi inverse and get a basis and I know that d phi is an isomorphism. Therefore, this will be continue to be a basis. This is very important and usually it is denoted, one omits this d phi inverse, etc.

Usually one just writes del by del x 1 etc. So, this is the usual notation for the basis. Of course one must keep in mind that these operators del by del x i are actually derivations defined on c infinity functions in r n. Not on the manifold but it is sort of cumbersome to write all this x lot of notation. So, one just writes it like this.

(Refer Slide Time: 15:34)



Now, let us move on to some examples of differentials and tangent spaces. So, as we have seen that first is the trivial case so if v is a vector space. Now, a vector space of dimension n, let us say, as such it does not have a topology but what we can do is let any basis of v will give an isomorphism to r n since it has dimension n. So, in particular we get a bijection from v to r n. Now, what one does is you just declare this the topology on v is just the pullback topology from r n. So, in other words declare a set to be open if and only if phi of that is open in r n.

So, you force this phi to be a homeomorphism and then of course, it becomes a manifold as well with a single chart namely the chart being phi itself. So, you get a manifold structured in a trivial way. What we have seen is that if so if u contained in v is open, then t p u is isomorphic to t p v which is isomorphic to r n. We really do not actually and in fact it is better to one can set up an isomorphism with v rather than r n.

So, the point is that one normally does not even though there is an isomorphism involved. One does not think of the isomorphism. So, in other words any tangent vector to u, we do not distinguish between a tangent vector to u and tangent vector to v.

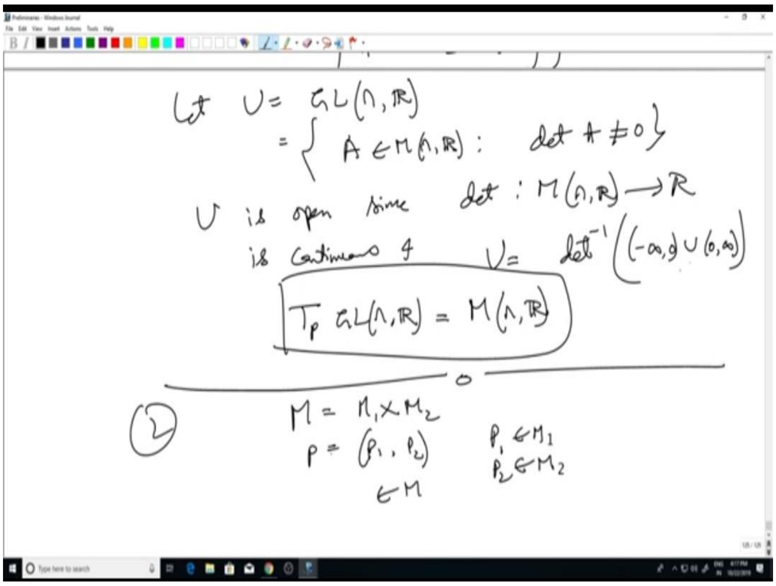
After all, we saw that the only difference is that tangent vector to u will act on all c infinity functions on u while tangent vector to v will act on all c infinity functions on, which are defined on v, the largest set v, but one, we saw that it one can go back and forth between the two because to see the action of a derivation on a c infinity function, all that matters is the value of the c infinity function close to the point p and because of that one does not distinguish.

Now, what about this? T p b is isomorphic to v. I do not want to go via r n to see this. In fact directly one can see the t p v is isomorphic to v and isomorphism is exactly the same as what we talked about earlier. So, v, so in order to see this I start with an element of v, capital V and I define a derivation so let f belong to c infinity v. So, you define v of f. The definition is exactly like in the Euclidian case d by d t f of p plus t v and then t equals to 0.

The good thing is that this f of p plus t v is just a real valued function of a real variable. One can forget about the intermediate vector space on this derivative is in the usual simple classical one variable calculations. So, and the point is that this gives an isomorphism between v and t phi v. I do not have to specifically mention r n in the process.

So, as specific examples of this, one can take v equal to the space of n cross n matrices m n r. This is isomorphic to the vector space r n squared and what all this shows is that if we regard m n r as a manifold it is just a vector space then the tangent space at any point, point m n r is a matrix. This just can be identified with m n r itself and it has an important open subset.

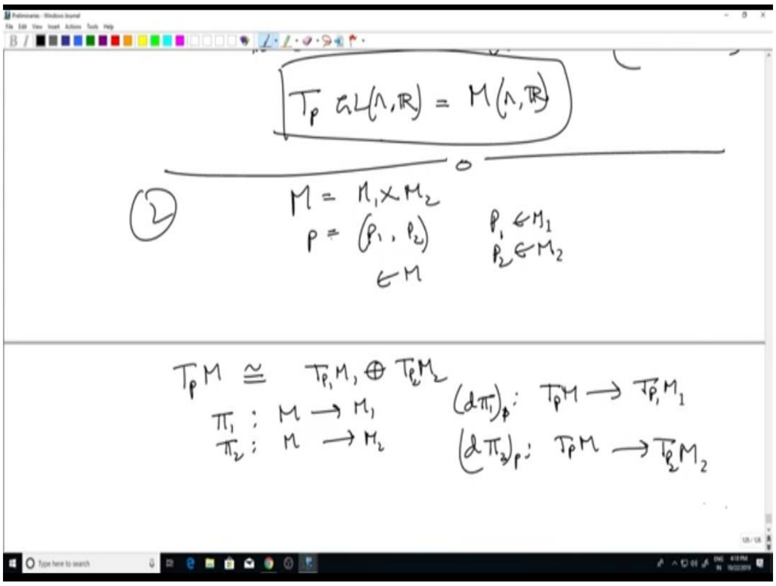
(Refer Slide Time: 21:30)



Let u equal to g l n r which is a set of matrices a in m n r such that det A is not equal to 0 and this u is open since the determinant function from m n r to r is continuous. In fact, it is just a polynomial function and u is nothing but so just did this function det inverse of the interval minus infinity to 0 union 0 to infinity. So, this is an open disconnected open sub interval of r. It is a continuous function so the inverse image of an open interval would be open.

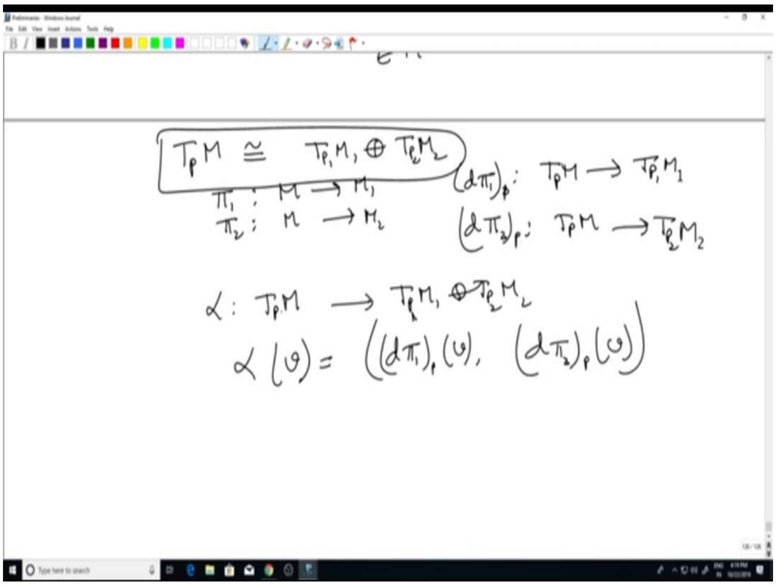
I get an open subset here and this is a manifold as well since it is an open subset of a vector space and our discussion about shows that the tangent space to any point to g l n r is in fact can be identified with the bigger vector space m n r. So, let us keep these two things in mind. So, the other example that I want to talk about is let us look at a product manifold and m 1 cross m 2, m equals m1 cross m 2. Let us take a point p. Well, a point p will have two coordinates, p equals p1 comma p 2, p 1 in m 1, p 2 in m 2. So, this point p is in m.

(Refer Slide Time: 24:29)



I want to look at the tangent space at t p m. The claim is that I want to claim that this is isomorphic to t p 1 and 1 direct sum t p 2 m 2 and this isomorphism can be described as follows. So, we have the projection maps. Pi 1 from m to m 1, pi 2 from m to m 2. If I look at the derivative of the projection d pi 1 at the point p, this would be a map from t p m to well it would be the tangent space at pi 1 of p. Pi 1 of p by definition is p 1 so it is t p 1 m and here it would be t pi 2 of p would be a map from t p m to t p 2 of m. The claim is that so if I combine these two projections, I will get this isomorphism that I am looking for.

(Refer Slide Time: 25:56)



So, I will define a map alpha from t p m to t p m 1 direct sum t p 1 m 1 to t p 2 m 2 as follows. Alpha of a vector v is I will just do d pi 1 at p of v d pi 2 at p of v. The claim is that this is an isomorphism and this is quite easy to see. It is actually enough to check that since I mean we already know that the dimensions are of the right sizes.

So, this note that dimension of the left hand side t p m is the same as the dimension of the right hand side. So, it is enough to check it is injective or surjective. So, or one can directly check it is both injective and surjective.

See, the thing is that so maybe I will leave it as an exercise to check that, one can check that, easy to check that alpha is injective and surjective. So, let me stop here. Maybe next time I might return back to this but we will stop here today. Thank you.