An Introduction to Smooth Manifolds

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Lecture No 11

Smooth Maps

Hello and welcome to the 11th lecture and the series, so I will continue with by giving some more examples manifold and move on to talking about smooth maps between manifold.

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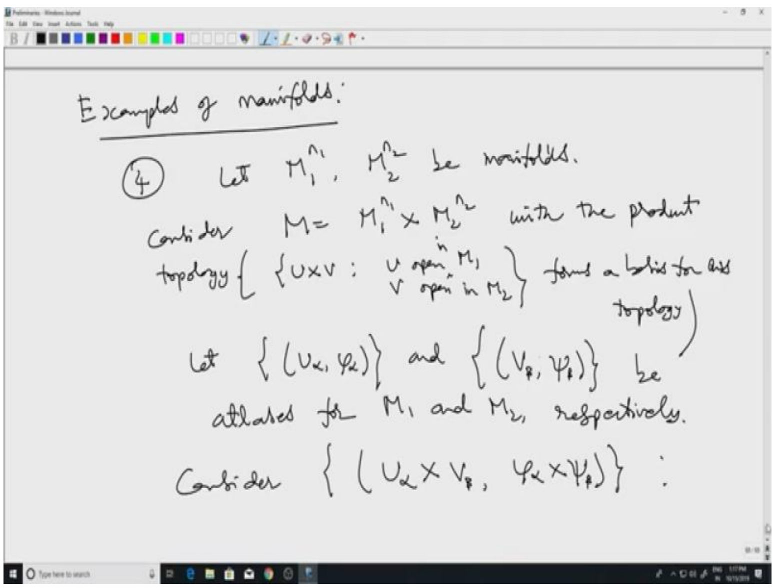

So let me start with the case of, so the fourth in my list last time I talked about, I ended with open subsets of manifold. Now the fourth in this list would be a product manifold. So, let M1 n1, M2 n2, be manifolds, then let us look at consider M equals M1 n1 cross M2 n2. So consider this space, well to begin with we have to put a topology on this or if we regard this M1 and M2 as metric spaces we have to put a metric, we can come up with the metric on this.

Now in terms of topology this would be with the product topology. So recall that the product topology its defining feature is that, if you look at open subsets of the form U cross , where U is an open subset of M1, V is an open subset of M2, then sets of the form u cross V forms a basis of the topology. So, the main feature is that sets of this form U cross V, U open in M1, V open in M2 forms a basis of this topology. So, that is all one needs to know about the product topology. And so now right, so this makes a topological space, now I want to define a manifold structure on this.

So the very definition of the product topology suggests that a chart on it M1 cross M2 cross should, it is a reasonable thing to start with a chart U in M1 and a chart V in M2 and look at the product. And, of course 1 has to write down a suitable homeomorphism on that chart, so let us do that. So let, I will start with the atlas on M1, U alpha phi alpha and V beta c beta b atlases for M1 and M2 respectively. Consider U alpha cross V beta, then I will define this map phi alpha cross C beta, where the maps are defined as follows.

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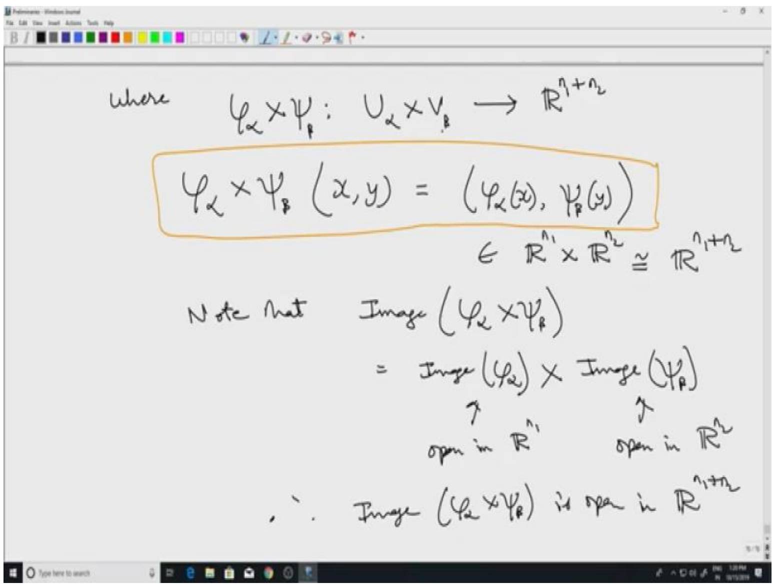

Where phi alpha cross C beta. Now this would be a mapping from U alpha cross V beta into Rn plus n1 plus n2 and this map is defined in a rather obvious way. So, I will take x, y where x is a point in U alpha y is a point in V beta and just define it to be phi alpha x C beta y. So, this phi alpha x would be a point in Rn plus 1 Rn1 and c beta would be a point in Rn2, so actually would be a point in Rn1 cross Rn2 which we naturally identify with Rn1 plus n2.

Note that so with this definition, note that the image of this map phi alpha cross C beta is just the image of phi alpha with the Cartesian product of this with the image of C beta. So, it is quite clear that this is the case just by the way the map has being defined here, just from this everyone can see immediately that the equality holds.

And, the point of writing this down is that image of phi alpha is, so this open in Rn1, this is open in Rn2. Therefore, the product, the Cartesian product of 2 this open set with this is would be an open subset of, therefore image of phi alpha cross C beta is open in Rn1 plus n2. Since it is an Cartesian product of open subsets of Rn1 and Rn2.

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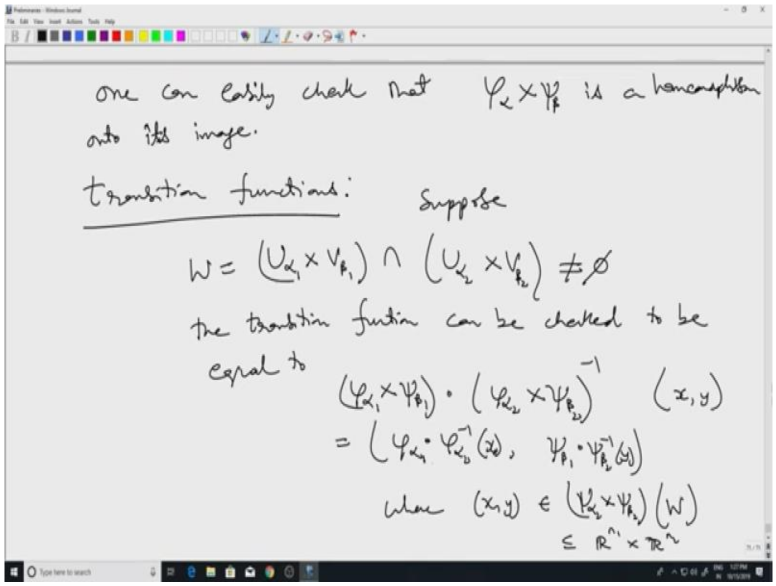

Moreover, one can easily check phi alpha cross C beta is a homeomorphism onto its image. So it is a continuous map and its inverses is continuous. And, that follows immediately from the corresponding properties for the phi alpha by itself and C beta by itself. Well what about the transition functions? It is a bit a messy to write down but anyway the formula is simple enough.

So, let say we have, suppose so as usual one has to start with 2 charts which intersect on M1 cross M2, suppose U alpha 1 and U alpha 1 cross V beta 1, this is 1 chart intersection. U alpha 2 cross V beta 2 is not empty, this is the case where transition function is even defined and it is, of course one has to write down the corresponding domain and so on.

But, the map itself is rather easy to write down so, the transition function can be checked to be equal to, so it is phi alpha 1 what I called phi alpha 1 times C beta 1 composed with phi alpha 2 times C beta 2 the whole thing inverse. And the way this maps are defined it is just phi alpha 1 composed with phi alpha 2 inverse, rather let me just act it on a point x, y. Now this time x and y would be a point in Rn1 cross Rn2 rather than the manifold.

Since the domain of a transition, function is an open subset of Euclidian space, so I have to start with x, y here, this n2 did not come out properly, so let me write it like this Rn maybe, let me rewrite this, so this Rn1 cross Rn2. So this is acting on x and then similarly C beta 1 composed with C beta 2 inverse acting on y, equal to this of course this x, y so I acted in x, y where x y belongs to, so strictly speaking it would belong to C alpha 2 cross C beta 2 of this intersection.

Let us just give it some name, let us suppose let us call it W, W equals this so it is just, then I can just write. So x, y belongs to this which is an open subset of Rn1 cross Rn2. The point is that the way these charts and chart maps were defined, it is quite easy to see that the transition functions are smooth because they sought of get decoupled like this, the x part gets separated here and y part get separated here and each map individually smooth.

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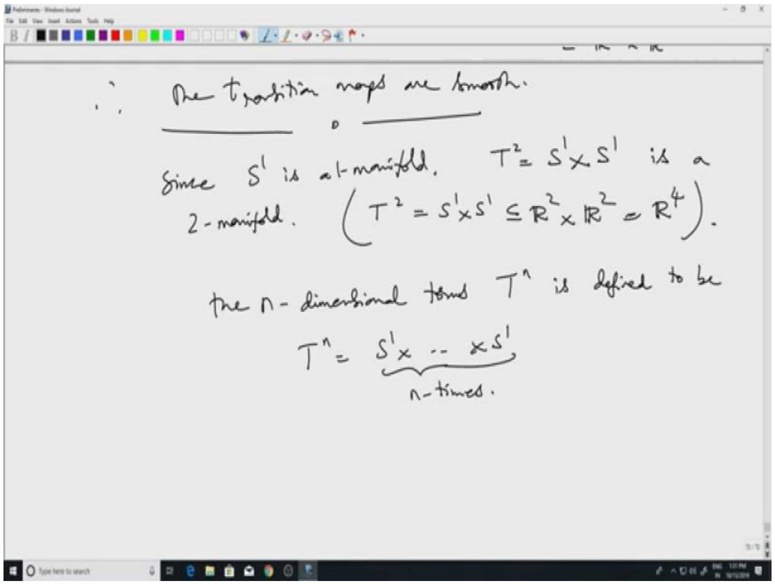

So therefore the transition maps are smooth. And that is, that pretty much concludes the list of simple examples that we have. But, already even with this rather limited list one can come up with some interesting manifolds. So for instance, since we know, since S1 is a manifold, T2 which is defined to be S1 cross S1. This is the 2 dimensional torus S1 cross S1, is a 2 manifold so this S1 is a 1 manifold so therefore this is a 2 manifold, a manifold of dimension 2.

So, this is a the 2 dimensional torus which is actually, now normally one thinks of a torus as a surface of revolution in R3. But, the way we have defined it, this T2 is actually a subset of, S1 itself is a subset of R2. So this would be R2 cross R2, which is R4 rather than R3. So, this description of the torus as a Cartesian product is as a subset of R4.

Now, of course one can prove that this is in some sense equivalent to the surface of revolution in R3, but we will return back to that when we talk about sub manifolds. Right now the 2 dimensional torus is a Cartesian product and there is no reason to stop at 2, the n dimensional torus, defined to be the n fold Cartesian product of S1. So and, this will give us an n-dimensional manifold. So but one can get far more interesting examples and the huge selection of examples.

Once I talk a bit about, first, I will start with smooth manifolds then tangent spaces then turn attention to sub manifolds. Then we will return back to more examples. Now let us start talking, talk a bit about smooth maps between manifolds, in fact, this is where when we want to define a smooth map. This is where the reason for defining a smooth manifold in such a complicated way in terms of transition functions becomes clear. It becomes clear why one had to go through the business of looking at transition functions.

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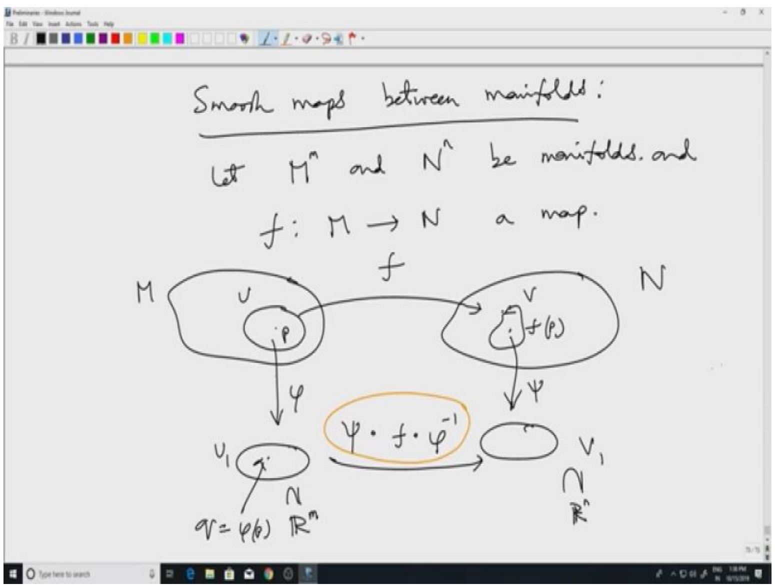

So let us start with the smooth maps between manifolds. So, I start with, let Mm and Nn be manifolds, as usual the superscript denotes the dimension. And f from M to N map. Let us, well, we want to talk about this such a map between 2 manifolds, talk about differentiability of such a map. The point is the only notion of differentiability that we know until this point is differentiability of maps, fresher differentiability of maps between open sets and Euclidian spaces. So ideally one would like to reduce to the case of Euclidean spaces and then using that one should be able to say that this f is smooth.

Now the way manifold is defined, if we start with a point P in N. I know that it is going to be contained in a chart U phi. So let me say, let me call this U1. Similarly, this f of P is a point in the target manifold N. It is contained in a chart as well so V and then a map C. So this is V1. So this U1 is an open subset of Rn and this is an open subset of R, like I said one would like to reduce to the case of map between open subsets of Euclidean spaces and then one already has a notion of smoothness and somehow use that to say something about f.

So this picture provides a way of reducing the to the case of open subsets of Rn, namely I, suppose I want to say that the map f is smooth at P, so I put it in some chart U and I put f of P in some other chart in some chart V. then I get a phi this whole apparatus comes along with a chart. Now what I can do is, I can look at phi inverse start with phi inverse then compose f and then do C. So in other words, phi inverse compose with f compose with C.

Now, we get what we want since this is now a map between open subsets of Euclidean spaces. So, it is natural to define to say that f is smooth at P if this corresponding map that I have here is smooth at, well P goes to some point here. Let us say this is q equals phi of P. So this corresponding map which I have here I would like to say f is smooth if this map the circle map is smooth at q. So in fact let us make that definition.

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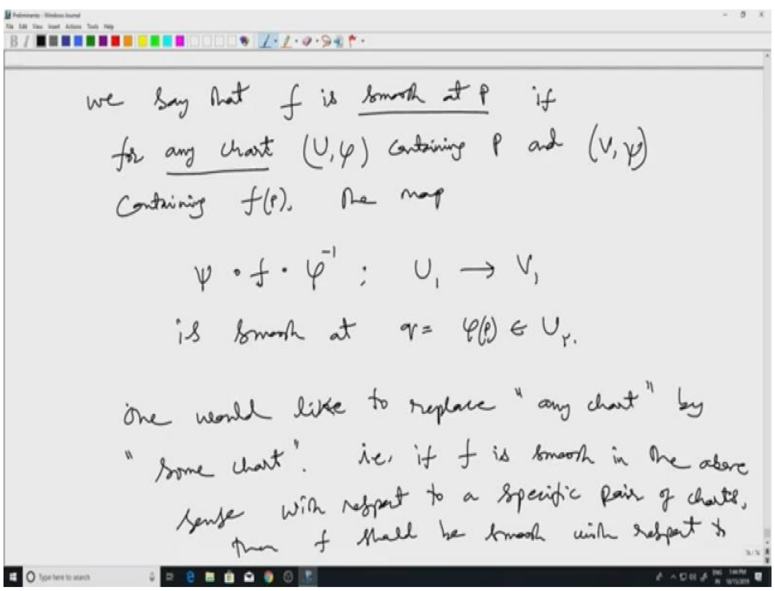

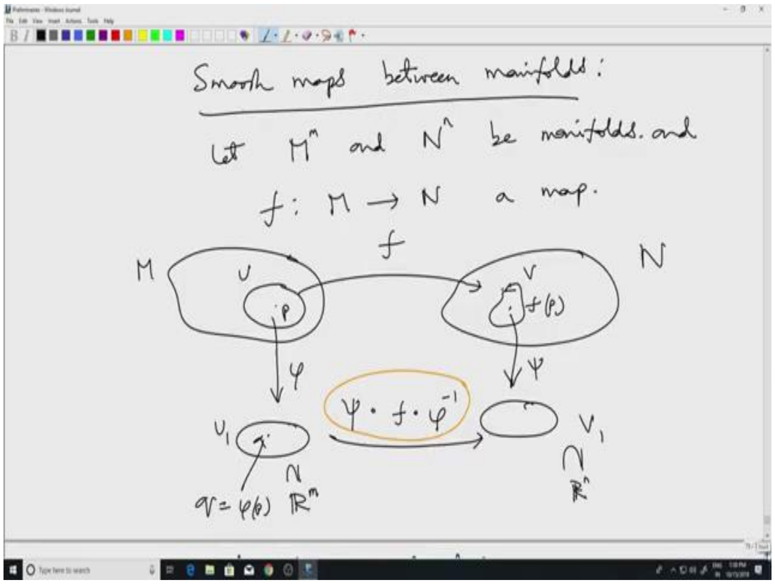

We say that f is smooth at P, if for any chart U phi containing V C containing f of P the map which I just wrote down there, which was, well, I first did phi inverse then f then compose with C inverse no not C inverse, the C this map from U1 to V1, so this U1 and V1 are open subsets of Rn and Rm. The map is smooth at q equals phi of P in U1. So this is the, so this now, so this is what we our notion of smoothness. Now the main point is as it turns out this business of here I wrote for any chart containing P and any chart containing f of P something should happen.

Well it might so happen that there are no charts I mean it might, it should not be the case that for some chart for a specific chart you U phi containing P and a specific chart containing f of P, this condition might hold that the corresponding map is smooth. But for some other chart it might not hold, one does not want that to arise. So, in other words, one wants to just be able to check this condition for a, rather than any chart one would like to replace any chart by some chart.

In other words, it should be if it this condition holds for 1 chart then it should be the case that it holds for all charts. Why do we insist on that, well it is just that when we are talking about a smooth manifold the charts themselves are not that important, even though the definition involves a chart, as in atlas, the atlas itself is not that important in the sense that it gives rise to a certain structure on M called, well which I did not emphasize but it might so happen that I will get, I can replace this atlas by another atlas which is in some sense equivalent to this. And when I do that essentially all the corresponding notions will turn out to be the same.

So, the thing is rather than emphasized charts, so whatever definition we make ideally we would like to make them independent of chart. Here for instance of course I have used charts but at least I should ensure that it should not depend on the specific chart. It should be if it holds for 1 chart it should hold for any other chart as well. That is one thing and secondly in any case if one wants to check that the map is smooth it will be quite hard to check this for every chart. Typically, what one does is one choses some nice charts and then checks various things.

For instance, when dealing with the sphere, the stereographic projection gave rise to 2 nice charts, where formulas for rather simple and one would like to check something for instance a map on the sphere is smooth. Just by working with stereographic projections rather than some other chart. So, for these 2 reasons that one would like to replace any chart by, if it holds i.e. if f is smooth in the above sense with respect to a pair of charts to a specific pair of charts then f should be smooth with respect to any pair of charts.

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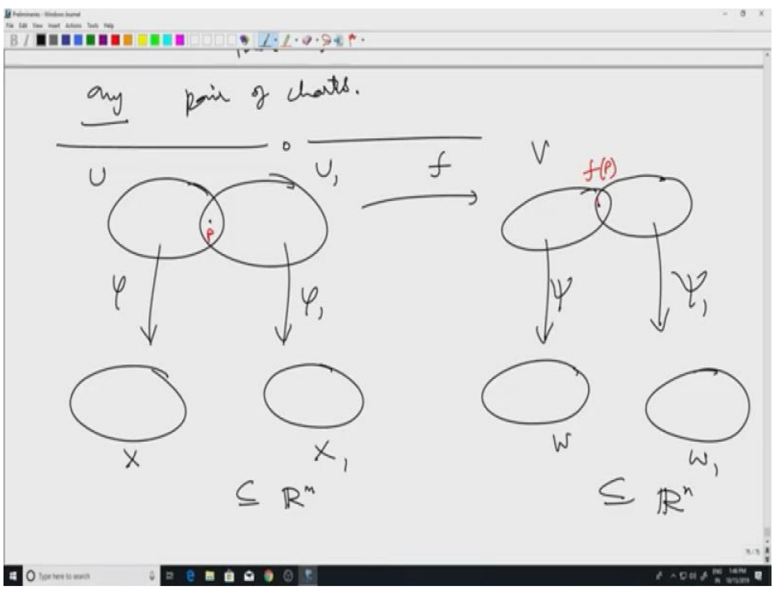

And, in fact that turns out to be the case so this the fact that this holds is precisely where one needs a transition function smoothness condition. So let me sought of illustrate why this is the case. So let us start with the point P so this is my point P and then let me call this as so I change notations slightly now.

Earlier U1 was an open subset of R1, R some Euclidean space but now I will use U and U1 to denote 2 charts on M. and the corresponding homeomorphisms phi, phi1. So this will land up in some X, X1 so both of these are, now these both inside Rm and then on the others so f is a my map and then on the other side I have V so this is V.

And, then, here this point is f of P and then I have maps C then C1, W, W1. So, these 2 are open subsets in Rn. Well, so let me stop at this stage then next time I will continue with this picture that I have and clearly going to details about where exactly the transition function condition plays a role. So let us stop here, Thank you.