Lecture - 35
Lyapunov Function Continued

Welcome back again. So, in the previous lecture, we were discussing now function and then Lyapunov theorems on stability and instability. As we have seen, the key idea there is to choose a positive definite function, which has required property for stability or instability as the case may be. So, it is very much dependent on the given system. So, one can easily construct positive definite functions, but that is not enough.

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So, we have to construct that positive definite function in order to have a stability or instability result and that in once he given system of differential equations. And, now we will discuss briefly, how to construct some standard positive definite functions. So, this is and again linear algebra will help us. And, it is the quadratic functions, which are studied in linear algebra and great detail quadratic forms. And, there are very nice results, which give necessary and sufficient condition for a quadratic form to be positive definite.

So, let me just briefly discuss that, what is the quadratic form. So, in when n variables, so quadratic form. So, let me again the use the same letter v. So, this v x 1 x n, a i j x i x j.

So, i j equal to 1 to n. So, will not be writing this summation all the time, so I just use the summation. So, here a i j are real certainly v at 0, all the x i's are 0 and certainly that is 0. But, this also could be 0 at some other x i's, which is do not want, so you want this be to be positive definite.

And, then what are the conditions on a i j to ensure that the and note that the quadratic form is not changed if I write a i j or a j i. And, so this also becomes a i j plus a j i x i x. So, the idea of witting this thing, so we may assume. So, this is the key assumption without changing assume a i j is a j i. So, that means that is the matrix the coefficient matrix, a i j is real and symmetric. Because, we are only interested in real quadratic forms real and symmetric. So, this means the coefficients the matrix elements a i j are all real. And, symmetry means this a i j equal to a j i.

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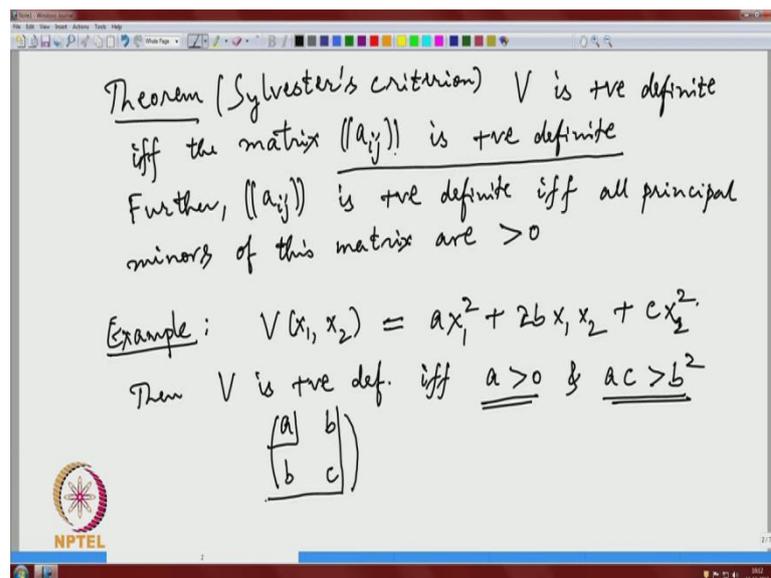
So, one of the key results in quadratic form, so let me just state as a theorem. So, this is due to Sylvester's it is just called Sylvester's criterion. So, V is positive definite, let me use that positive definite. If and only if the matrix, which is already real and symmetric is positive definite. And, actually this Sylvester's criterion describes when this matrix is positive definite, further a i j is positive definite. If and only, if all the principal minors of this matrix are positive.

So let, so will discuss this theorem in great detail, when you discuss the portions of linear algebra that are required for this course. So, let me just give an example here. So, this

what a this mean in two variables. So, in n equal to 2, I m just taking that. So, now I have just let me write that, so 2 b. This 2 is only for convenience c x 2 square. So, then V is positive definite, if and only if a is positive and a c is bigger than. So, in this case the coefficient matrix is just a b, b c and the principal minors are determinists of this 1 by 1, and 2 by 2 matrix.

And, that is what this criteria says, both the determinists are positive. So, a is positive and then the determine of the second principal minor, that is whole matrix that is a c minus b square that is also positive. And, that is says a c is bigger than b square. So, in practice what one does is choose such a quadratic positive quadratic function. So, that is all is possible by we can use the Sylvester's criteria. And, construct many quadratic forms, which are positive definite and then you apply this thing to the given system of differential equations. And see whether the second condition is satisfied for stability or instability. And then, the stability of that particular equilibrium point. In this case 0, we are transforming everything to be origin. And then we can decide whether that 0 is stable or unstable.

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So, let me explain through some examples and before going to example. So, again let me stress that, it is not always guaranteed that this quadratic form, will be enough to judge the stability or instability of a particular given system. One may have to go to higher

degree polynomials, so that we will see in some examples. So, the first example, so consider this 2D system x 1 dot equal to minus x 2 cube and x 2 dot equal to x 1 cube.

So, it is very similar to the simple harmonic motion we have already consider, expect that instead of in the simple harmonic case these are linear, but now here we have cubic non-linearity. So, you can check easily that 0 0 is the only equilibrium point in this case of this system. And, if you calculate the Jacobian of this system at this critical point it is 0 0. So, it is just 0 matrix. So, we have already seen in the study of linear systems, when the coefficient matrix is singular then that situation is described as degenerate system.

So, the vectors in the kernel of that matrix, which is not even now, so it is non zero. So, all will be solutions and in this case it is since this is 0 matrix, any point in r 2 is a solution of the linear system. So, that is in some sense totally degenerate. And, now let us go back to the non-linear system, so looking at this cubic power let us try this. So, consider this function V x 1 x 2 is equal to x 1 4 plus x 2. So, since it is fourth powers both variables. So, it is easy to check that V is positive definite is 0 only at the point x 1 equal to x 2 equal to 0 otherwise it is always positive.

So, that is one criteria is of this satisfy. And, let us calculate in order to decide the stability or the stability of the origin. So, let us conclude this, so compute. So, in this case it is easy, so grad V dot f. So, f is the right hand side here, so again let me write it. So, this is x 1 and I have minus x 2 cube plus d by d x 2 this is x 1 cube. So, if you compute this derivative of V with respect to x 1 and x 2. And you observe that this is 0.

So, the stability theorem Lyapunov is require that this to be less than or equal to 0. And, here it is identically equal to 0, so it is more than that. So, therefore, 0 0 is stable. And, since it is not strictly less than 0, it is not asymptotically stable. So in fact, so this analysis tells us that.

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So, in fact, the orbits of the system orbits of the given system are given by x 1 4 plus x 2 4 is equal to constant. So, let me write that constants in these are both positive. So, that also is positive, so let me write that as c 4 for c bigger than equal to constant. Of course, when you see c is equal to 0 we get the equilibrium 0.00 when c is positive. So, we get these orbits, so orbits satisfy this equation, so as an exercise. So, you just for c positive show that these are closed curves, just like circle these are not circles something different from that closed curves surrounding the origin.

You also can use, if you are familiar with mat lab or mathematica to draw these curves for different value of c. So, just this is not a circle, but I just it is not something like that, something this is origin something like that not exactly that. So, you can also do that and see that they are all like that. So, concept requires again just like circles, but something different on that. So, now, let us go to the second example, let me go to new page.

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So, this is your 3D system, so let me just write it x 1 dot is equal to minus 2 x 2 plus x 2 x 3, x 2 dot is equal to x 1 minus x 1 x 3, the x 3 dot is equal to x 1 x 2. So, these examples will tell you, the systems can have very different behavior they, though they might look somewhat similar, but you will see the difference. So, here again the origin is an equilibrium point. So, there are other equilibrium points, I will come to that in a minute.

So, just certainly, if I put all x 1 x 2 x 3, so the right hand side vanishes, you can easily check that equilibrium point and d f we compute the Jacobian at this point. So, again it is 3 by 3 matrix now. So, this is some computation, you have to do then how to do that. And, the Eigen values of this matrix are 0 plus or minus 2 i. So, again one of the Eigen values is 0 that means, the determinate is 0 this matrix is singular. So, again it is a degenerate system, so linearized system it is degenerate system.

So, all the vectors in the null space of this matrix are solutions of the linearized system. And now let us try, so since we are now familiar with quadratic functions, so consider this.

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So, V x 1, x 2, x 3 a quadratic path with only diagonal elements, c 3 x square. So, V will be positive definite. If all c j's are positive, so that is you can also check by the Sylvester's criteria. So, this is only diagonal matrix, so there is no problem. So, if all the c 1 c 2 c 3 are positive, then this is a positive definite matrix. Now, let us compute, because that is what is the requirement of Lyapunov theorems? So, little algebra if you do it. So, let me write it. So, c 1 minus c 2 plus c 3, x 1 x 2 x 3 plus minus 2 c 1 plus c 2 x 1 x 2.

So, if you look at this terms x 1 x 2 x 3 and x 1 x 2 no matter what is the constant size choose either to make them this coefficient positive or negative whatever. Because, it involves the product of this x 1 x 2 x 3 and in any product of the origin they can take both positive signs and negative signs. I will not be able to make either this if I want a non zero 3, non zero coefficients here, so it is very difficult to make this grad V dot f either a positive thing or a negative thing.

So, the easiest things, since it involves this products of x 1 x 2 x 3. So, easiest thing is to make this identically 0. So, that can happen only if c 1 minus c 2 plus c 3 is 0 minus 2 c 1 plus c 2 is 0. Remember c 1 c 2 c 3 are at our disposer. So, we can choose them, as long as they are positive that is fine. And now these are second condition, so there are two equations and 3 n o. So, you have 20 of choices, so one choice is choose c 1 equal to c 3 equal to 1 and c 2 equal to 2.

Then we have a positive definite function, for which grad V dot f is identically 0. So, again in the like previous case, so therefore, 0 0 the origin in 3 D is stable, but not asymptotically it. And, in this case, so because this is identically 0 we also see that the orbits are given by x 1 square plus 2 x 2 square plus x 3 square is a constant. So, let me write that as c square. Again, when c is equal to 0, we get the equilibrium point when c is positive, so these are all ellipsoids.

So, here much easier to recognize then the previous example, here it was x 1 4 plus x 2 4 equal to c 4, but this only quadratic terms. So, we know this what this represents, so these are ellipsoids again surrounding the origin. There region for meet stretching this we will see later, why I am stretching that, that is a precisely point in Poincare–Bendixson theorem.

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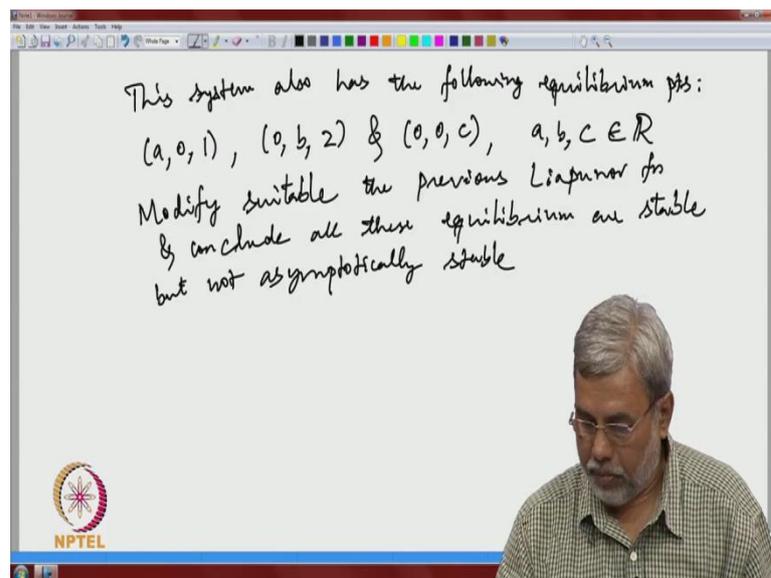
So, in this system also has the following equilibrium points there are infinitely many in fact. So, let me write few of them, so this a 0 1, 0 b 2 and 0 0 c, so these are all equilibrium points, where a b c are arbitrary real numbers. So, we have constructed Lyapunov function at the origin, and you can simply modify the same thing to construct Lyapunov functions for any of these equilibrium points. And you see that, so modify suitably the previous Lyapunov function. And, conclude all these equilibrium points are stable, but not asymptotically stable.

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So, in the next example, let us so it is modification of 2. So, again the same system just in perturb it little bit more minus 2 x 2 plus what was it x 2 x 3. And now, I perturb it x 1 cube and x 2 dot again same thing as before, again perturb it by cube same thing with third equation again I perturb it by cube. So, you see that the difference between the previous example and this example is that the system is perturbed by this non-linear terms.

And, now check this lit again bit algebra check that 0, 0, 0 is the only equilibrium point. So, only in this case there are no other equilibrium points. So, little algebra 0 is certainly an equilibrium points at 0 these right hand sides vanish, but it is the only point, where the 3 equations on the right side vanish. And again the Jacobian is the same, so there is no change in the Jacobian 1 0 0, 0 0 0. So, again it is a degenerate system, there is absolutely no change.

So, again let us start with that from the previous example let us consider this, which has helped us in the previous example. So, again x 1 square plus 2 x 2 square x 3 square, so it is a positive definite function. And now, let us compute grad V dot f with respect to the system, previously you constructed, computed del V dot f for the previous system.

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And now, grad V dot f, so please do it some algebra x 3 4. And now, we see that this is less than 0 for all x not 0, when one of them x 1 x 2 x 3 is not 0, then, it is strictly right. So, this is a negative definite function, so therefore, again by the Lyapunov theorem we conclude that, 0, 0, 0 is asymptotically stable. Now you see, just a perturbation of a system can change the nature of the equilibrium point. So, previous example it was only stable, but not asymptotically stable.

And, in this example we have perturb the system by some cubic non-linearity. And, we now obtained that, this origin is asymptotically stable. So, again if you change the sign of that perturbation, instead of minus x 1 cube minus x 2 cube minus x 3 cube, if you put plus you see that 0, 0, 0 is unstable. So, a slight perturbation can change the nature of the equilibrium point. That, is an important point to be observed, because in practice we never get a system exactly it is always an approximate system.

So, would like to know, what perturbations due to the equilibrium point. And, this is a subject of an advance subject, which is called the structural stability. So, we are not discussing in this course, so it goes into that. So, you are perturbing the system itself and asking, whether that system is stable or not, so that is a another topic. So, let me go to the next example now, so this is a 2D example. So, instead of x 1 x 2 let me write x dot y dot, so this is 2 x y plus x cube y dot equal to minus x square plus y 5.

So, I will not discuss the linearized system, so you can do it yourself. So, again 0 0, so, you can check 0 0 is the only equilibrium point. So, here also the Jacobian at this point of this system 0 matrix. So, let me not again and again discuss that, so let us try directly go to this non-linear system. And see whether this origin is stable or not and again for that thing. So, consider this positive definite function V x y. So, I m using the notation x y instead of x 1 x 2, so x square plus 2 y square, so this is a positive definite function.

Then no doubt about that and you see why that two is chosen. So, according to the system 1 has to construct this Lyapunov functions. They are not we have to construct this things and if you work out this quantity grad V dot f in this case you find it twice x 4 plus 2 y 6. And, because of even powers this is positive definite and therefore, by the Lyapunov instability theorem, so 0, 0 is unstable. So, we also have some instability examples.

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So, the next one is again it 2D system, so this is fifth, so 2D system. So, let us consider the system x 1 dot is equal to minus x 2 plus x 1 x 2 minus x 1 cube minus half x 1 x 2 square. And, x 2 dot minus 3 x 2 plus x 1 x 2 plus x 1 square x 2 minus half x 1 x 2 square. So, 0 0 is certainly a equilibrium point and I leave it to check whether this system has any other equilibrium point. And study if there are any and what is their stability, whether they are stable or unstable etcetera, you can check that.

So, here again if you compute the Jacobian this will be 0 minus 1, 0 minus 3. So, again the Eigen values are 0 and minus 3. So, it is again degenerate, because the matrix is singular again it degenerate system. So, now, just try this, try V of x 1 x 2 is equal to half for convenience we put it 3 x 1 square minus 2 x 1 x 2 plus x 2 square. So, if you use Sylvester's criterion, you can easily check that this is positive definite. And since, we are just in two variable and a quadratic function we know how to complete this square.

So, you can also see the positive definiteness in much easier way in this case. So, this is just 2 x 1 square plus x 1 minus x 2 square. So, both these terms are non negative. So, certainly at x 1 equal to x 2 equal to 0 it is 0. And, suppose this is 0 then 2 x 1 square that is x 1 0 here is x 1 equal to x 2. So, again we get that x 1 equal to x 2 equal to 0, so it is for x 1 x 2 not equal to 0 V is always positive. So, either you can use Sylvester's criterion or this simple computation to see that is positive definite, that is the first job we have done.

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And now, we have to compute del V dot f compute. So, this one is let me do that thing, so this del V by del x 1 into f 1 plus del V by del x 2 into f 2. So, in this case, so let me just write it, you can just compute it. So, this is 3 x 1 4 some algebra plus 2 x 1 square x 2 minus 2 x 2 square. And this, you can write as minus 2 x 1 4 plus x 1 minus x 2 square x 1 square minus x 2 square plus x 2 square. And, all the three things in bracket are

nonnegative and it is 0 only if x 1 equal to x 2, so this is negative definite. So, therefore, again by Lyapunov theorem 0, 0 is asymptotically stable.

So, you see in the so for all our examples, you also observe that the equilibrium point in question is not hyperbolic. So, the linearized system always has an Eigen value either a 0 Eigen value or Eigen value with real part 0. So, these are all non hyperbolic, so they cannot be the stability or otherwise cannot be judge by perron's theorem or Hartman grobman theorem. So, here you see the usefulness of Lyapunov function and Lyapunov's theorems on stability and instability.

So, that you should bear in mind, so all our examples defeat that thing. So, all the equilibrium points in question or not hyperbolic, so that you there in mind. Final example, so, this is so there will be more x assign exercises in the assignment sheet. So, you can practice this construction of Lyapunov functions forgiven systems. And, do not think that this in for example, in this previous example (( )) taking this, this has not just come by you know just like that. So, it is different constants have been put and tried.

And the main objectives to have a definite sign for del V dot f. So, then the suitable constants have to be chosen. So, here I am writing final thing, but these constants have come after some trial and error, that is you should bear in mind. So, it is not just automatic. I given a system I cannot immediately see why this 3 minus 2 and 1 should come no, no that is that will not happen. You have to try and important thing is we have to have a sign for this grad V dot f. so that you should there in mind.

And now, one more example x 1 dot, so this is somewhat special. So, let me spend some time on this x 1 dot is equal to x 1 square plus 2 x 2 5. So, again it 2D system, x 2 dot is equal to x 1 x 2 square. So, observe that if this 2 x to the 5 this perturbation we are not there, this is already an unstable system. And, if this x 1 were replaced by some positive constant again the second's equation is also unstable. So, we can expect this 0,0 is the only equilibrium point. We may expect for example, here if x 1 is negative term this becomes the for second equation it becomes a stable. So, you may expect that, the 0,0 is unstable, but we cannot immediately see that.

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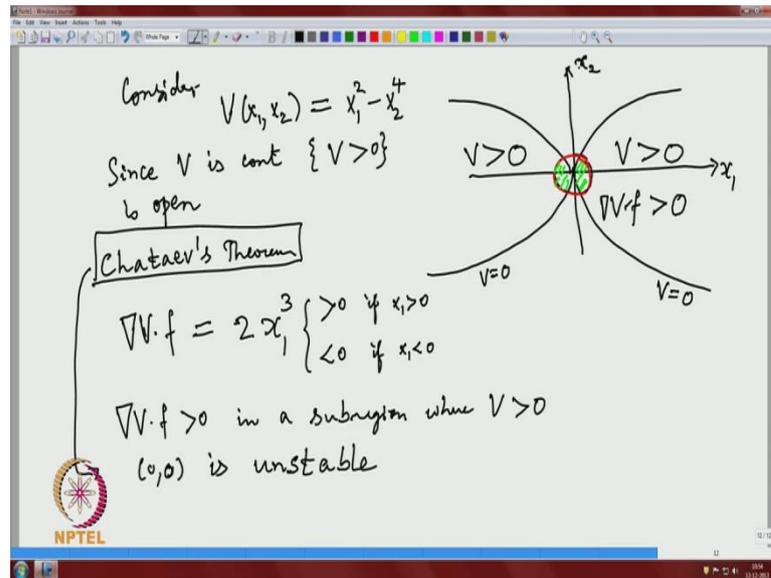
So, let us consider this V x 1 x 2, x 1 square minus x 2 4. And, because of this negative sign certainly V is not positive definite. So, you can have this V x 1 x 2 equal to 0 for x 1 square equal to x 2 4 and x 1 x 2 need not be zero. However, if you draw this graph in x 1 x 2 plane sort of parabola, but not exactly parabola. So, it is something like that it is symmetric. So, these are all the points, where v is 0, v 0 and inside this V is positive here. So, every neighbor hood of the origin so may be just use this. So, every neighbor hood of the origin contains a small region, where V is positive.

So, since V is continuous certainly this, this so since V is continuous the set where V is positive is open. So, it is so if it contains a point then it contains also a small ball around that you can see here clearly in the picture. And now let us compute, so there is so recall I made a comment in the sketch of the proof of instability results. So, observation by chetaev, so this is also called chetaev's theorem, so recall that already done it. So, this hypothesis of chetaev's theorem in that every neighbor hood of the origin contains a region where v is positive, which is clearly here.

So, this if I can make it this is the region, where it is positive. And now, let us compute grad V dot f. So, simple calculation, you see that it is 2 x 1 cube which is positive for x 1 positive. So, this is so this whole thing positive if x 1 is positive and negative, if x 1 is negative. So, if you recall the second condition in chetaev's theorem, what we require is this grad V dot f positive in the region or a sub region in a sub region, where V is

positive and certainly here we have though immediately see that. So, this so in this region this grad V dot f is positive.

So, now you apply chetaev's theorem and conclude that 0, 0 is unstable. So, it is indeed our initial guess is right and 0, 0 is unstable and remember here the hypothesis of Lyapunov theorem is not satisfied Lyapunov theorem requires this grad v dot f to be positive definite everywhere in a small neighbor hood. But, chetaev observation was that it is sufficient to assume the positive definiteness in a sub region, where v is positive. So, 0, 0 is unstable in this case. So, please go through all these examples, these are very important for general understanding of this stability analysis. And next time we will discuss 2D systems.

Thank you.