Ordinary Differential Equations
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Lecture - 34
Lyapunov Function

Welcome back. So, we will continue the discussion in the previous class on stability analysis of non-linear autonomous systems. And, last time we saw that at the hyperbolic equilibrium points, the linear stability analysis, implies the non-linear stability analysis. So, may now concentrate on non hyperbolic equilibrium points. And, this will be handled by the use of Lyapunov function. It is a simple tool, but very powerful tool as we see it. So, before I go into the details.

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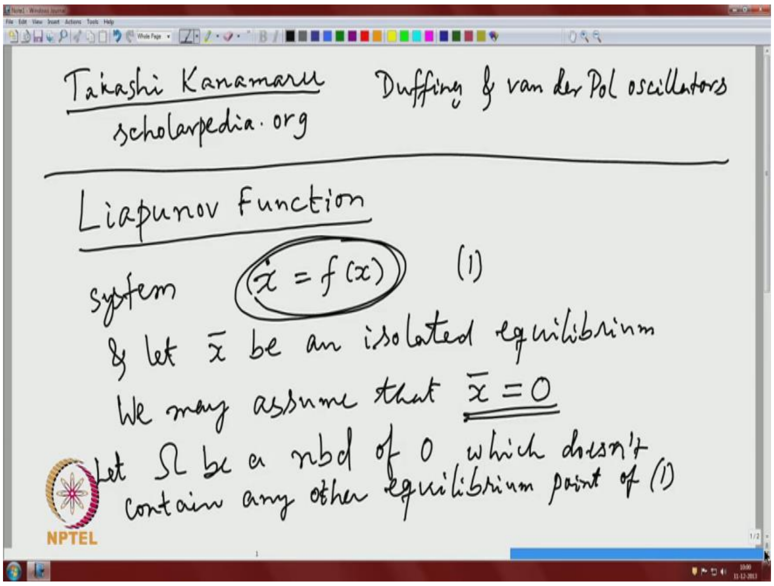

So, let me just mention a reference Takashi Kanamaru, there is a nice write up by Takashi Kanamaru on duffing and van der pol oscillators. So, this is available on the web it is a scholarpedia dot org. So, you can get more information on this duffing and van der pol oscillators, last time we discussed little bit about that. So, you can refer to this, write up by Takashi Kanamaru. So now, we move on. So, introduce Lyapunov function definition is very simple that is not.

So, again consider the autonomous system x dot equal to f x. So, this is our standard system. So, system and let x bar be an isolated equilibrium point. So, again will be studying in the stability of orbits of this system, around this equilibrium point. So, we may assume, so this is our here after we do this, just for some simplification in writing. We may assume that x bar equal to 0. This we can always achieve by translating this system into a neighborhood of x bar, you can just check that. So, that is not this is no general the loss of generality. So, just stick to this thing. And let omega be a neighborhood n b d neighborhood of 0, which does not contain any other equilibrium point of 1. So, this is possible, because where assuming 0 is isolated.

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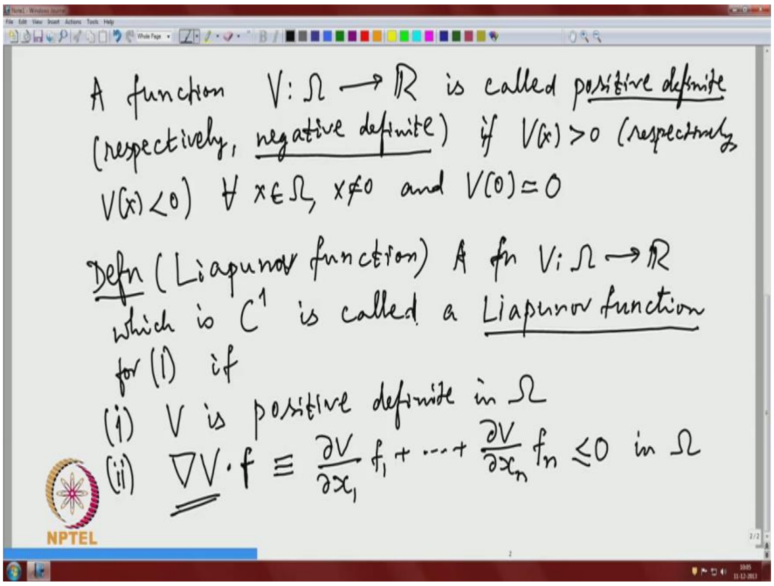

So, a function so this is a definition a function. So, all our analysis now restricted to that neighbor wood omega. So, we define functions and other objects on that omega. So, V from omega to R. So, real valued function defined on that, neighbor wood of the origin omega is called positive definite respectively negative definite. So, any real valued function defined on omega respectively negative definite. So, this is definition. If V x is positive respectively, V x negative for all x in omega and x naught 0. And V of 0 is 0.

So, this is I am not putting any other restrictions like continuity. And, other thing that those things will come as you go on. Now, this is just this is a real valued function V takes only positive value accepted these 0 or negative value if it is negative definite. So, now definition of the Lyapunov function. So, this is very useful tool. And, it has been even adopted in other situations, a function V from omega to R which is C 1. So, it is continuous differentiable and derivatives also continuous, all partial derivatives is called Lyapunov function for the system 1.

So, that is so that is where the system now enters, otherwise just a serial valued function right. So, what is the connection with one. If, so two conditions let me write one, V is positive definite in omega, so that is the previous definition. And second one that is connection with the system 1. So, del v dot f. So, del v is gradient of v, so this is by definition. So, this is f is coming from the system. So, this is nothing but del v by del x 1 into f 1 plus etcetera.

So, we are in n dimensional situation. So, remember V is a function from omega to R so it is a function of n variables. So, now, this is a real valued function, for all x in omega, so this is less than or equal to 0 in omega. So, at all points in omega, we require that this gradient p dot f. So, this is scalar produce and that is what I am defining. So, this is a vector and f is also vector grad V is a vector. So, this is the vector del v by del x 1 etcetera del v by del x n.

And f is the vector f 1, f 2, f n. So, I am taking the scalar product of those two vectors at every point in omega. And we require that this is less than or equal to 0. So, that is where the given function is related to the system via the second condition.

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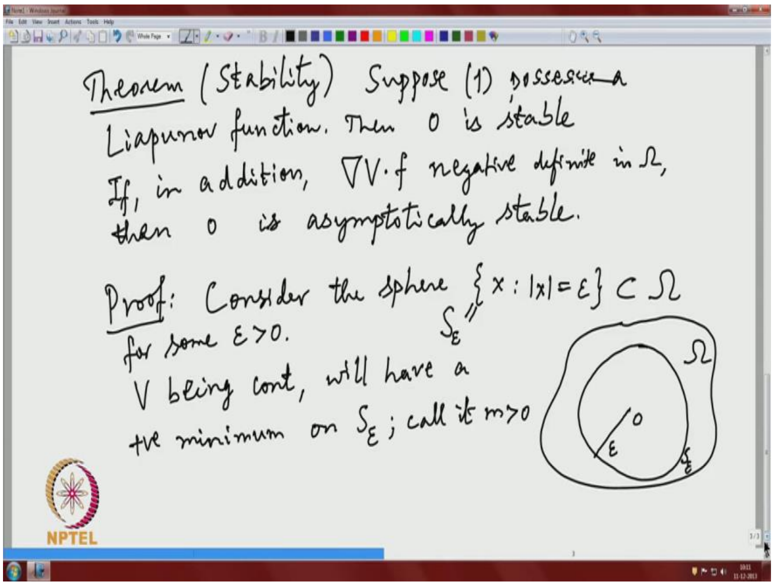

So, now theorem, so these concepts were introduced by Lyapunov, this is stability theorem. So, suppose 1, the system 1 possesses a Lyapunov function. Then, 0 is stable. So, remember again by our notation 0 is and equilibrium point of 1. And, so as an when the system 1 possesses a Lyapunov function 0 is stable. if in addition, so now, we strengthen little bit the second condition grad v dot f. So, again look at the definition grad v dot f.

So, that is again a function defined on omega to the real number. So, this is a real valued function defined on omega. I want this is negative definite in omega. Then, 0 is asymptotically stable. So, now you clearly see, what all the Lyapunov function place as for as stability and asymptotic stability of any equilibrium point of the given system place. So, that is the important of Lyapunov function. And, proofs are very easy and they use simple calculus.

So, not very deep analysis, but you can already see, the inter play between analysis and the differential equation. So, I have stress this point again and again in an many situations. And, here is another situation, where you see that inter play very clearly. So, proof, so again you go back to the definition of stability and asymptotic stability I will not recall that here. So, consider the sphere, so this is in n dimension sphere set of all x in omega, mod x is equal to epsilon.

So, let me just y ((Refer Time: 14:01)) a let me dot put set of all x. And since, omega is a neighbor wood of the origin, so I can choose small epsilon, such that this is in omega for some Epsilon ((Refer Time: 14:27)) is 0. So, to speak, so here is a simple diagram I will put omega here. And, this is the origin. So, I am putting this ball in. So, you will be working in that particular neighbor wood.

So, V being continuous, will have a positive minimum this is the, because V is positive definite by definition will have a positive minimum. So, let me call this S Epsilon some notation on S Epsilon call it m, so m is positive. And it is the minimum, so this is S epsilon.

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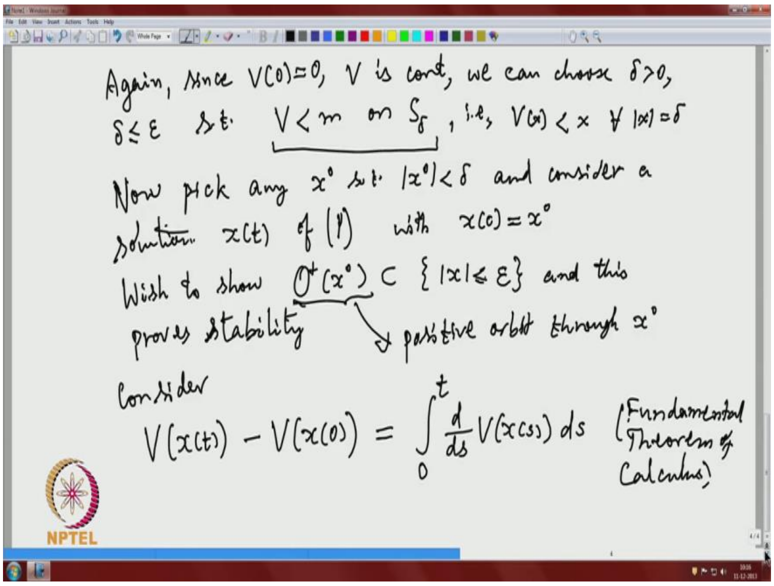

So, again by continuity again since V 0 is 0. V is continuous. So, we can choose delta positive and delta less than or equal to epsilon, such that V is strictly less than m on S delta. So, let me again stress this thing, what does this mean, so this means that is V of x is strictly less than x for all mod X equal to…. So, that is let us go back to the previous page ((Refer Time: 16:57)). So, this so delta will be somewhere here, so this is delta and I am choosing delta, so that the V will be strictly less than and remember m is the minimum of V on the bigger sphere mod x equal to epsilon.

Now pick any x 0 such that mod x 0 is strictly less than delta. And, so you start your orbit and consider a solution x t of 1 our system with x 0 equal to…. So, I am now starting the orbit in the ball of radius delta. And would like to show that, this orbit through x 0, remains within the x Epsilon and that prove the stability. So, wish to show, so the r positive orbit is contained in mod x less than or equal to epsilon. And, this proves stability.

So, implicitly I am assuming that the solution x t of 1 with x 0 equal to x 0, exists for all t positive. So, this is a standard assumption that I have been telling, so that is then would like to show that the positive orbits. So, this is remember positive orbit through x 0. So, that would like to show. So, once we show that thing, that is prove this stability for this thing consider V of x t, so now, we are bring in the and V of… So, you compose the solution x with this real valued function V the Lyapunov function V.

So, how you get another a function, from the real line into the real line. So, that is the composition of V and x. And, this by fundamental theorem of calculus, one variable calculus, so this is 0 to t d by d s of, so this you already seen many times. I am just applying that is. So, this is fundamental theorem of calculus, so this is a remember this. So, just, you see how simple things in calculus are utilized. Now, the second hypothesis come into picture.

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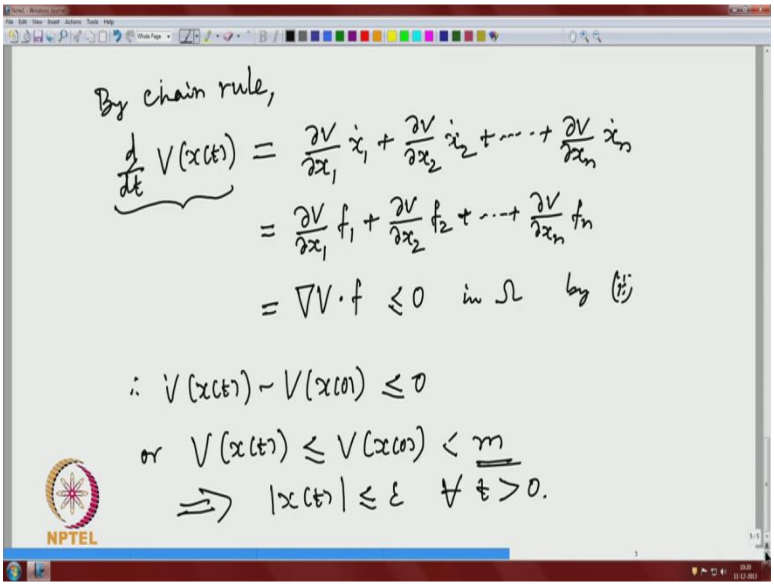

So, by chain role this is another thing from calculus. So, d by d t of V of x t is nothing but d v by d x 1, x 1 dot plus d v by d x 2, x 2 dot plus etcetera, plus d v by d x n, x n. But, remember x is a solution of system 1. So, this is nothing but d v by d x 1 f 1 plus d v by d x 2 f 2 etcetera. And, this is f n and this is nothing but del v dot f that is what we defined and by our assumption this is less than or equal to 0, in omega by condition 2. So, now if you put back this is remember in the integrant we have this expression.

And, now by chain role I have computed this one and shown that is less than or equal to 0. And, again now you go back, so here you see, being the integrant I have that derivative. So, that will be, so therefore, V of x t minus v of x 0 is less than or equal to 0 by that integral or so this v of x t is less than or equal to v of x is 0. And, remember x 0 is chosen within that, ball of radius delta and by our choice this is less than m and this is minimum on the sphere mod x less than or equal to m.

And, that proves that mod x t is less than or equal to m and Epsilon for all t. And that completes the problem. So, that so stability is one part. So, stability is easier.

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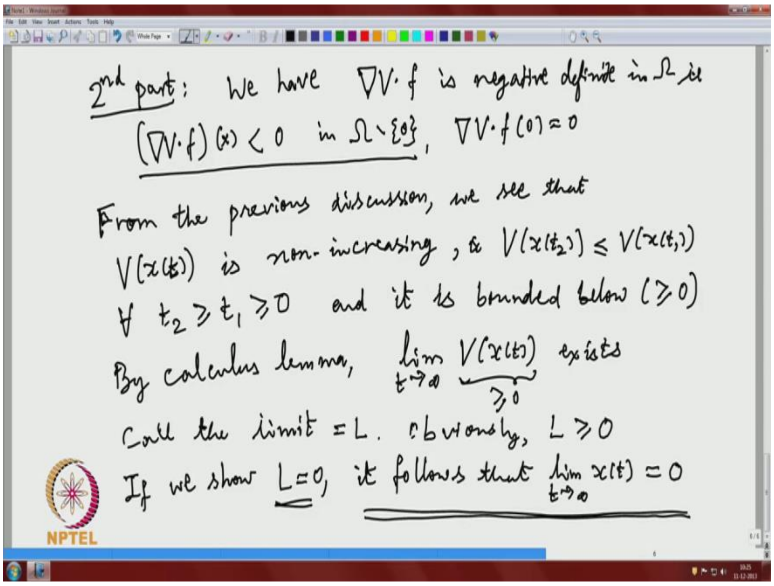

So, now let us go to the second part, second part, so we have the additional assumptions. So, we have grad V dot f is negative definite in omega. So, again let me stress, so that is grad V dot f at a any point x less than 0 in omega minus 0 and x 0 t 0 f 0 is. So, this is important, this is we are going to use now. So, again from the first part, from the previous discussion, we see that this function, this composite function v of x t as function as a function of t, is non- increasing.

So, again you can just use that, fundamental theorem of calculus you integrate between any two times t 1, t 2, t 2 bigger than t 1. And, you see that, is in non the non increasing. So, that is V of x of t 2 is less than or equal to V of x t 1 for all t 2 bigger than t 1 ((Refer Time: 26:59)). So, we are just concentrating on that positive orbit, here that composite function is non increasing. And, it is bounded below the function is bounded below. Why bounded below, because that is our assumption it is bigger than or equal to 0.

So, here we have a non increasing function, which is bounded below and by calculus lemma this limit V of x t, t tends to infinity exists. So, here I have a non increasing function V of x t, which is bounded below, that is why this limit exists. And, since this is a non negative function, call the limit, equal to L. So obviously, L is greater than... So, use to show that L equal to 0. So, if L equal to 0, you immediately you see that since V 0 is 0and v vanishes only at 0.

So, if we show L equal to 0, it fallows that limit x t, t tends to infinity should be 0. Otherwise, if this limit is different from 0, then you go back here. So, limit of as t tends to infinity of V of x t, will be V of this limit, whatever that limit. And, that to prove, if you are going to show that, that is going to be 0, then that will get a cancel. So, this is important. So, once we show that the limit, this limit L is 0, it fallow the automatically that, this limit of x t is 0. Otherwise we get a contradiction.

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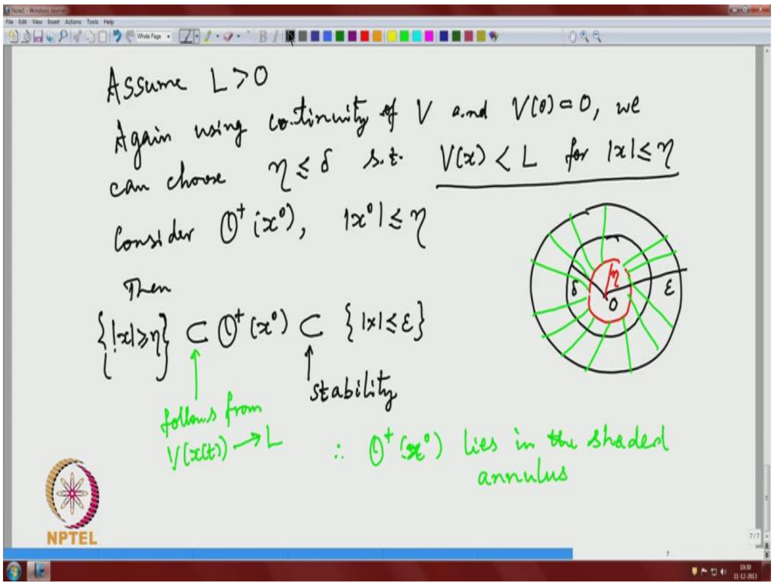

So, assume to get a contradiction, assume n is positive. So, again using continuity of V and V 0 equal to 0. So, these are simple arguments, we should check them using your knowledge of calculus. We can choose, eta less than or equal to delta, such that V of x will be strictly less than L for mod x less than or equal to eta. So, here is the picture. So, we have this big, let me again recall that. So, this is the epsilon. So, this is origin and we have a delta here in a side. And now, I am going to choose, may be let me just. And, remember this eta is chosen show that, we have this condition.

So, this can do, because I am assuming L is positive. And, will show how we get contradiction on that. And now, we consider this, this x t consider the orbit, positive orbit starting at x 0. And now, I take mod x 0 less than or equal to eta. And, which is automatically in that delta, so that is no. And, by assumption consider this thing, by assumption and this construction, you see that then O plus x 0 the entire orbit is contained in mod x less than or equal to epsilon.

So, this comes from stability, so this is part one, so no problem. And now, the second part, this set of mod x. I want to show that it is bigger than....So, to what I am claiming is the orbit lies in this green. So, it will not go to 0. And, this part comes from, comes follows from that V of x t converges to L. So, it can go beyond it. So, the entire orbit lies in this annulus. So, therefore, O of x 0 lies in the shaded annulus. So, follow this geometry annulus. So, now we use the second hypothesis now.

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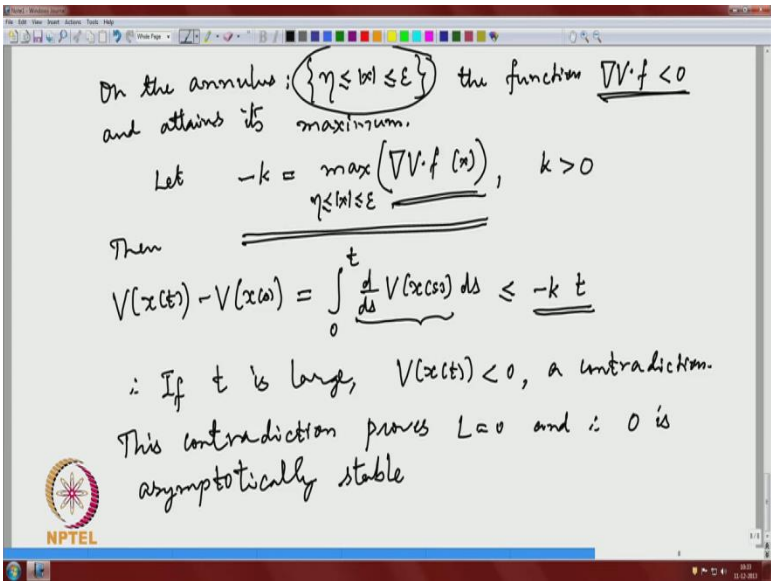

So, on the annulus, so that is a compact set on the annulus what is the annulus. So, eta less than or equal to mod x less than or equal to Epsilon is closed and bounded annulus. So, that is compact set the function, so this is the second assumption grad V dot f is negative and attains it is maximum. So, I want to put a notation for that. So, let minus k, so since it is negative, this is negative, so I just put maximum of grad V of f at x. And, x I am taking V this. So, that is the continuous function.

And this is a compact set, so it will have a maximum and since it is negative. So, this k is strictly positive. And, now go back to the equation, which is given by the fundamental theorem of calculus. Then V of x t minus V of x 0 is equal to 0 to t, d by d s of v of x s d s. And, this we have already computed this is del v dot f x and now I know that, this orbit remains only in this thing, so this is just less than or equal to minus k. So, this is where I am using this definition of k. And x the orbit is strictly to only this annulus, all these facts are used here.

And, this I am replacing it by it is maximum, which is minus k and integral 0 to t d s is just ((Refer Time: 37:09)) the. So, therefore, so remember this is k is positive, so as t becomes large, so this is an negative number. So, I can always exceed this x 0. So, if t is large, V of x t will be negative. And, that is a contradiction, because V is s positive definite point. So, this contradiction proves, true L equal to 0. And therefore, 0 is asymptotically stable.

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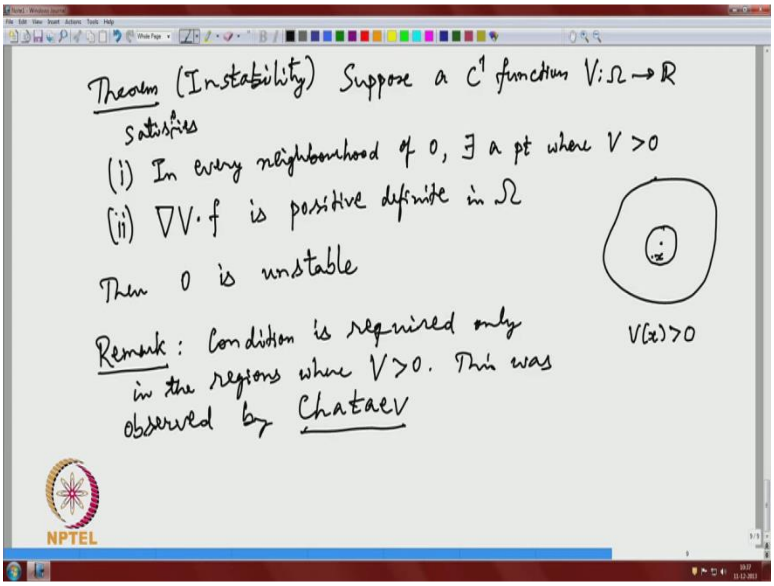

So, Lyapunov also has a theorem for the instability results. So, let me state that also. So, theorem again instability, so suppose a c 1 function. So, this is Lyapunov function if you want to call it for the instability case a c 1 function V from omega to R satisfies the following satisfies 1. In every neighborhood every ball every neighborhood of 0, there exists f a point, where V is positive. So, I am not requiring V to be positive in entire neighborhood, but an every neighborhood there is a point, so just let me again I will show you every neighborhood.

So, there is some point call it x, so where V x is positive. I am not requiring that V to be positive, everywhere in that neighborhood. And, the second condition, that again connecting the function with the system our system is positive definite. So, later on we will see that, this condition do not require ((Refer Time: 40:49)) no omega. We do not require this positive definiteness in everywhere, require this positive definiteness only when in the region, where v is positive then o is unstable.

So, remark, so I will just write that, whatever I said now condition 2 is required only in the regions, where V is positive. So, this was observed by Chaadaev Russian names, so I may have might have miss spelt it but this is. And, there are other as for as instability results are concerned, there are more relaxed conditions, given by others. So, this is sufficient for many purposes.

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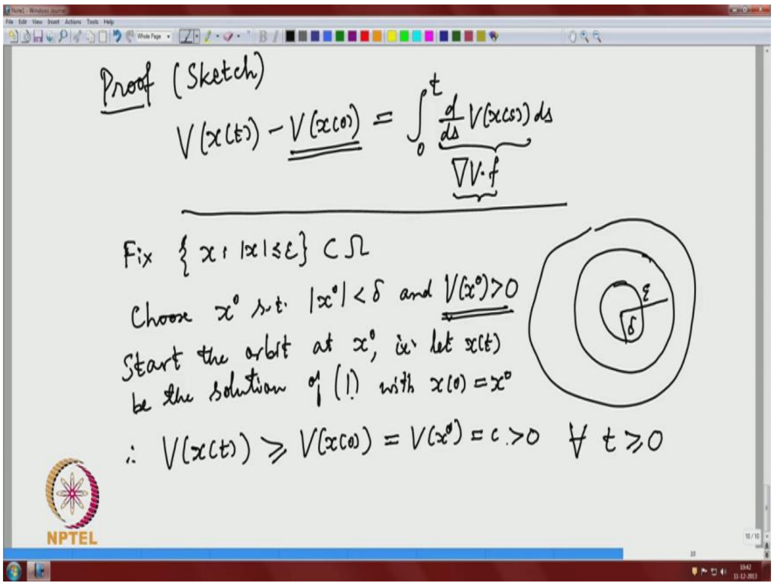

So, let me sketch a proof of this thing and we will consider the examples next time, so sketch. So, again the starting point is the fundamental theorem of calculus. So, d by d s of V of x s d s. And, this we have already seen that, this is del V dot f evaluated at x s ((Refer Time: 43:04)) that is fine. And now, you see this our assumption is that, this is nonnegative. So that means, now we will be increasing. So, that is the idea, so I am just sketching it. And, you start a solution where this is positive.

So that means, if I start at a point where v is positive, then I remain always bigger than that by this. So, this is an important thing for us. So, let me just sketch it. So, again you start with, so fix one ball fix is x such that mod x less than or equal to Epsilon and this is in our omega. So, this omega there and this is x Epsilon and now I take another delta here. So, by hypothesis, so choose x 0, such that mod x 0 is less than delta. And, V of x 0 is positive.

And this is possible by our assumption on V. So, in any neighborhood, there is a point where V is positive. So, this is the condition 1. Start the orbit at x 0. So, that is so this same thing as saying. So, let x t be the solution of our system 1 of 1 with x 0 equal to x 0. So, then by hypothesis second hypothesis, so therefore by the integral a fundamental theorem of integral calculus. And, the assumption we see that, this is bigger than or equal to V of x 0. So, this is V of x 0 and call, this c and this is positive. So, this is true for all t greater than or equal to0. So, that is important. So, if start the orbit at such a point, then V along that path remains bigger than this c.

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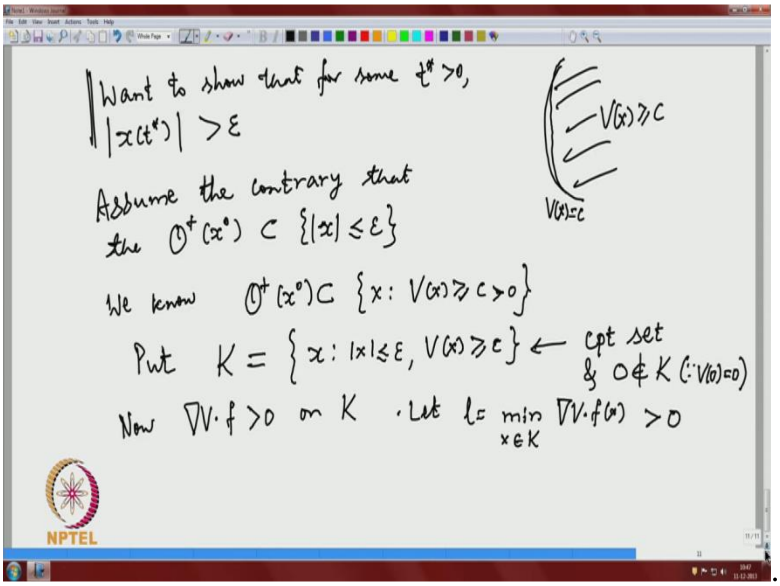

So, let me draw again a diagram. So, that is, so now, I have this ((Refer Time: 46:36)) this is V of x equal to c. And, I want to remain, so, this is just V of x bigger than or equal to c. So, that is the reason and part of omega everything in part of omega. So, the orbits is confine to this region. And that is what, that estimates shows, now would like to show that, that orbit has to leave the Epsilon ball at later stage.

So, want to show, that for some let me use t star, t star positive this x t star leaving the Epsilon ball means, this bigger than want to show that. Assume the contrary again. So, this is a sketch, we have to fill in the details using the simple analysis assume the contrary. That, the orbit the positive orbit starting at x 0 is contained in this x less than or equal to Epsilon assume that. I already know that. So, we know that, this orbit x 0 is already confine to this one.

So, set of all x such that V x is bigger than c remember this is positive. So, in particular this, this is positive. And now, we are assuming that the orbit is also contained in this ball, so it is in the intersection. So, let me put some name for that. So, put K is equal to this set of all x, such that mod x less than or equal to epsilon. Let the same time I want this c. So, by using continuity of V and this compactness of this epsilon. So, you see that, this is a compact set. And, important thing is 0 does not belong to K, because V 0 is 0. So, if I have forgotten that, to include that thing. So, include that V 0 is 0 that is all with it V 0 is 0.

So, since it is positive there. So, V 0 cannot be here. So, again by continuity, I will come to that later. And now, you recall that hypothesis. And, since now I am, so you see that hypothesis, the second hypothesis that grad V dot f be positive definite, in entire omega is not required, I am just applying only to this set. So now, grad V dot f is positive on K. And K being compact set and this is 0 is not there. So, this so let l be minimum of grad V dot f. And now, you take x in K and this is positive. Now, we are more or less done, so again the same argument.

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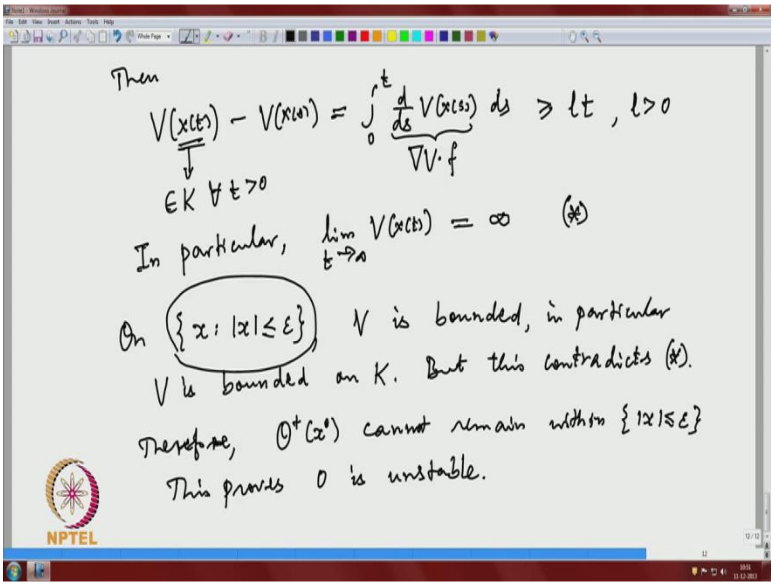

So, then V of x t minus V of x 0 again this is equal to 0 to t. So, I am writing it again and again, so that you realize it is. Usefulness, and now this is grad V dot f. And, we are assuming that our orbit is restricted to K. So, this is in K for all t bigger than 0. So, that is important assumption. So, that this, whatever value we are taking now there all in K and in K, so this is always bigger than or equal to l t. So, l is positive remember, l is the minimum of this grad V dot f.

So, in particular, so this proves ((Refer Time: 52:40)) limit V of x t at t tends to infinity this. So, does not see anything about the x t as now, but now you see on the set x less than or equal to epsilon. So, this is the compact set in the euclidean space r n this is a compact set. And, V is continuous V is bounded. So, in particular V is bounded on K, because K is a subset of this bounded on K. And, if I call this, but this contradiction start, a bounded function cannot have any infinite limit this contradicts.

So, why this contradiction the contradiction comes, because we are assuming the orbit is restricted to. So, the Epsilon ball in this ball the we are assuming that thing. And to get this contradiction therefore, so the orbit therefore, orbit cannot remain within this boundary. So now, you again go back and see the definition of the instability and this proves, this proves 0 is unstable. So, you see that, the question of stability or instability in this method of Lyapunov can be analyzed by the construction of suitable Lyapunov functions.

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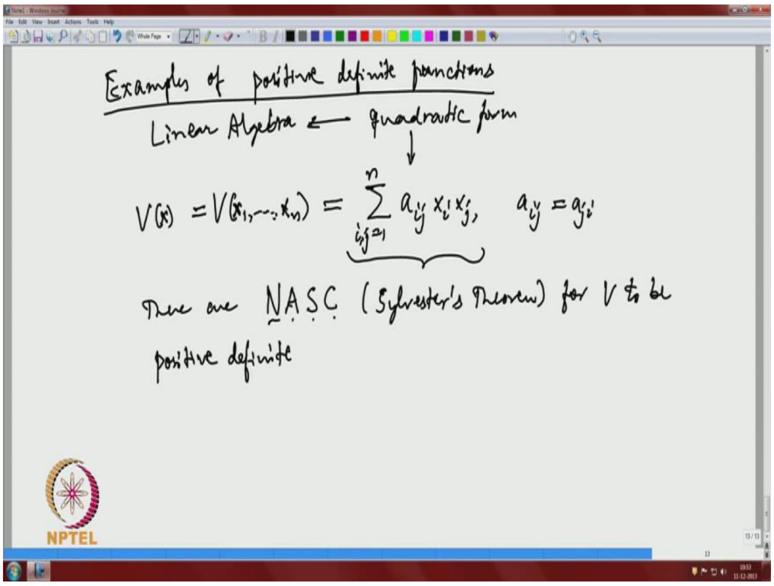

So, again where are the examples, this we will discuss the even in the next class. So, examples of positive definite function or negative definite function. Because, we are not seen any this for and again these come from linear algebra. We will see in the next class. So, they come from linear algebra, what that it forms. So, a quadratic form is of the form.

So, this is V of, so in n variables x 1, x 2, x n. So, that is V of x is given by a i j, x I x j. So, a i j are real and symmetric.

So, these are termed as quadratic forms in n variables. And, there are well definiteness necessary and sufficient condition for this V to be positive definite. So, there are necessary and sufficient conditions. So, these are this is Sylvester's theorem in the theory of quadratic forms for V to be positive definite. So, we will discuss this. So, these are some important class of quadratic positive definite functions.

And, for a suitable for a given system, we have to choose a suitable quadratic form of course, not all the time this quadratic forms will works, so sometimes we have to even take higher order polynomials. But, in many cases just this quadratic forms will satisfies and we will see some examples in the next class. So, we will discuss examples of positive definite functions Lyapunov functions and other things in the next class.

Thank you.