Ordinary Differential Equations
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Module - 6
Lecture - 32
Stability Equilibrium Points Continued

Welcome back. So, in the previous class we were discussing about Stability of an Equilibrium Point of our autonomous system. And if you look at the definition, so there is a need to isolate our equilibrium point. So, let me start with that thing.

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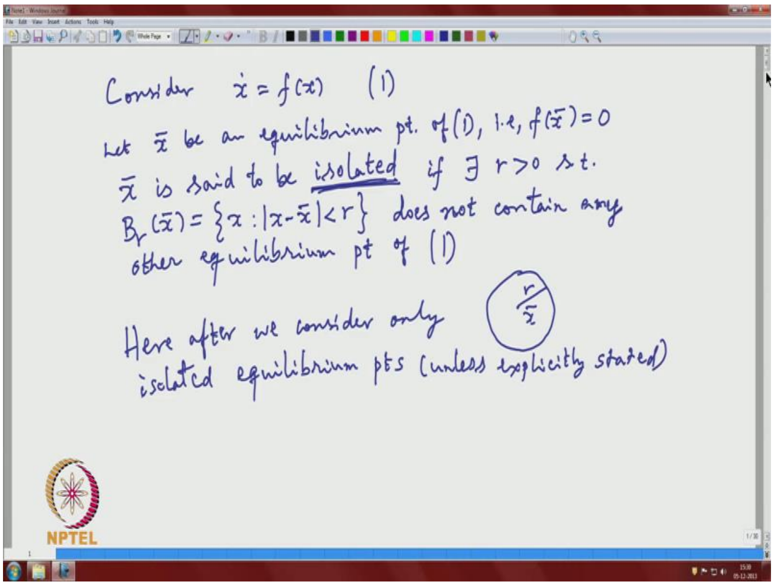

So, consider, so this I always refer to system 1. So, let x bar be a equilibrium point of 1. So, again let me remind you that is, it is the right hand side is 0. So, if there are n equations here, so it is x bar is common 0 of all those functions.

So, x bar is said to be isolated equilibrium point. So, in most of the examples we consider they are all isolated. There will be some examples where the equilibrium points will not be isolated. We will indicate those things. If there exits r positive such that, so I introduce a notation here. So, B r x bar, so this is what we call open ball centered at x bar with radius r. So, this is set of all x in r n such that whose distance from x bar is less than r. So, this is the open ball does not contain any other equilibrium point of 1.

So, geometrically I just want to draw that ball. So, this is x bar. So, this is, so in 2 dimensions you can imagine this is really a ball. So, this is and this interior should not contain any other equilibrium point of 1. The reason for this thing; since, we are in the definition of stability you recall. We want to start a solution very close to x bar and we want to remain there. So, if there are other equilibrium points then that solution may go to tend to that other equilibrium points.

So, you are just wants to avoid, so we just isolate it. So, that is why this isolation. So, so once this, we take x bar to an isolated point. So, you reframe the definition of stability and asymptotic stability of an isolated point. So, I will not repeat these things. So, here after unless, here after we consider only isolated equilibrium points of course unless explicitly stated. So, this is usual before we move on further. So, immediately let us consider again those examples we did in the last class.

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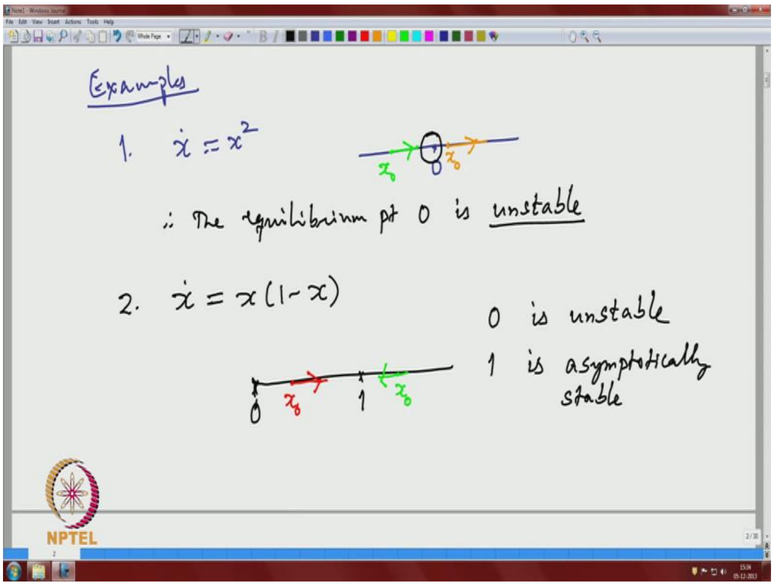

So, examples, so again start with 1 d x dot equal to x square and 0 is the only equilibrium point here. So, this is 0 and certainly it is isolated, because there are no other equilibrium points, so much more than isolated. And we saw, so if we start a solution with initial value on the when x 0 is positive. So, it will move in that direction and if we start a solution in x 0 negative, then it will move. So, this we have already worked out.

So, if I start the solution in a neighborhood of 0 and left side, it approaches zero. But right side it goes away from 0. So, therefore the equilibrium point, the only equilibrium point 0 is unstable. Just look at the definition again. So, for stability we want the solution to remain all the time in that neighborhood. But here on the right side of the origin in this case, the solution moves away from the 0. So, that is why it is unstable.

The second example logistic model. So, normalized several things. So, just we, so here the equilibrium points are 0 and 1. So, obviously they are isolated. So, I do not have to stress that thing 0 and 1. And again we have done the analysis. So, if we start a solution between 0 and 1. The solution the orbit moves towards right and it in fact approaches 1. And if we start the orbit through x 0 which is bigger than 1, then it moves towards 1 again.

So, looking at this picture and just again look at the definition of the stability instability. So, 0 let go back to blank. So, 0 is unstable and 1 is asymptotically stable. So, we can see when there are more equilibrium points. So, the nature of one equilibrium point may be different from the other one within the same system. So, for example, see this. So, one is unstable and another equilibrium point is asymptotically stable.

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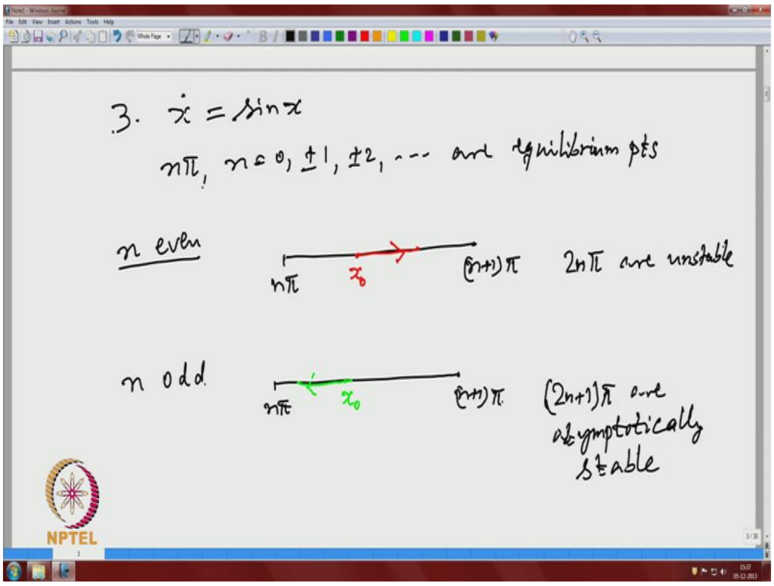

So, now let us move to that the one dimensional cosine of the pendulum equation. So, here x dot equal to sin x. So, we did this in great detail in the morning. So, in fact one can get explicit solutions. So, that was an exercise. So, here n pi n equal to 0 plus or minus 1 are equilibrium points. And do you saw, the nature of orbits changes with n being odd or even. So, let me just consider the case of let me write separately. So, n even that is the case.

So, let me just draw here n pi n plus 1 pi. So, if I start orbit through a point here, in this interval open interval then the orbit moves towards that. And n odd same thing n pi n plus 1 pi. And now, I start the orbit from x 0 here and it is moves towards that. So, you see that this when n is even, so this 2 n pi are unstable and 2 n plus 1 pi. So, these are all odd multiples of pi are asymptotically stable.

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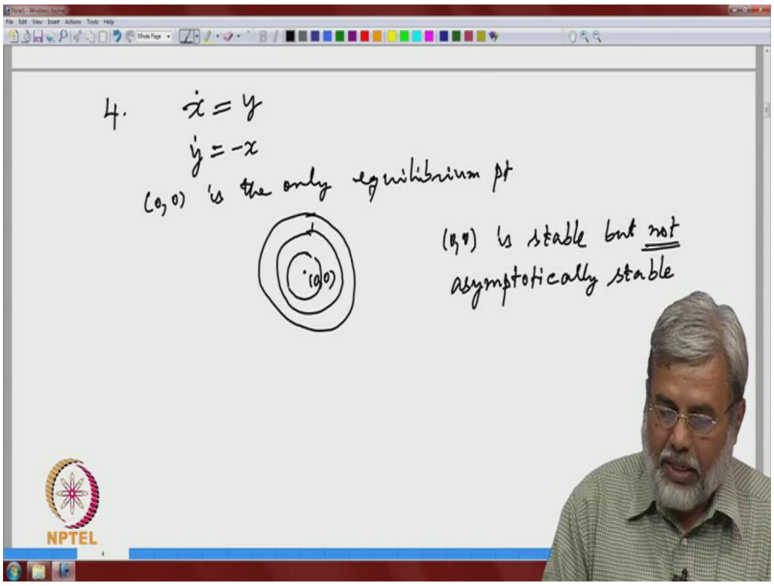

So, finally one more example. So, pendulum with small oscillations. So, this is the approximation, so 2 D. So, this is 2 D example. So, we saw in the previous lecture the orbits are nothing but circles. So, here again 0 0 is the only equilibrium point. So, it is isolated naturally. And all our orbits, so this is 0 0 are circles. So, if I start an orbit, so there are all concentric circles. So, though it has come here, so it is concentric. So, no doubt if you start in a small neighborhood around origin, the orbit the positive orbit stays close to an origin there.

But in any case it will never it can never converge to the origin as t goes to infinity, because it is always lies on their circle. So, here we have an example this 0 0 2 D case is stable, but not asymptotically stable. So, this is a very typical example and we coat it at many places not asymptotical. So, stability does not automatically imply asymptotic stability, but asymptotic stability requires stability. So, asymptotic stability is something more than stability.

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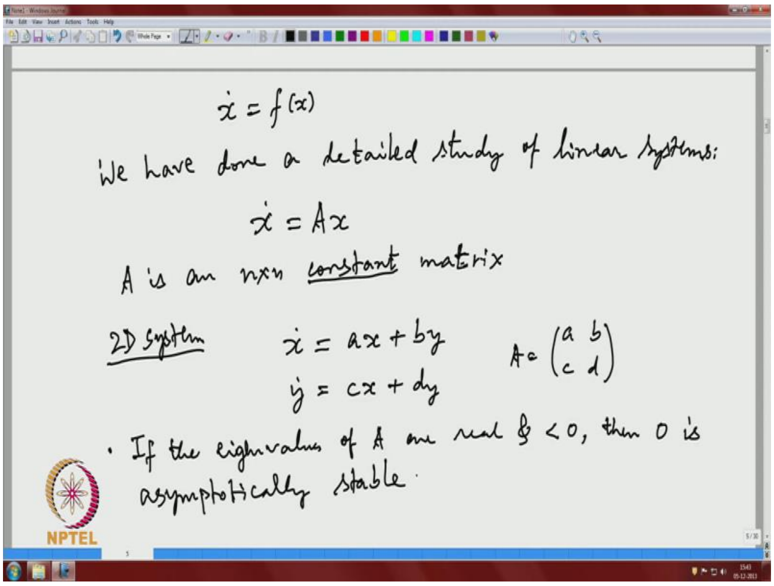

So, now we have defined stability and asymptotic stability of an isolated equilibrium point. But then how do I decide. So far, so I have worked out few examples that is fine. But suppose I want to decide, when a given equilibrium point will be stable or unstable or asymptotic stable. What should I do? So, that is next question and we have already done enough of this in a special case.

So, that is our starting point. So, we have done a detailed study of linear systems that is our guide impression. So, let me just write that. So, that f x a special thing here, so a x. So, a is an n by n constant matrix. So, it entries do not depend on t. So, because we want a autonomous system constant matrix. Let me just recall what we have done there. So, let me just instead of writing a general system let me just write a 2 D system.

And recall some of the facts we did. So, in this case we have only two unknowns. So, a x plus b y and y dot equal to c x plus d y and this a will be in this case just this 2 by 2 matrix. And we have also learnt that how the nature of Eigen values of this 2 by 2 matrix has played an important role in the description of the trajectories orbits of this system. So, let me just recall that.

So, if so let me just recall this, if the Eigen values of a are real and negative less than 0 then. So, you recall for this linear system 0 is a equilibrium point when the determinant is not 0, 0 is the only equilibrium point. So, in case we are assuming Eigen values are both negative. So, there are non zero. So, a is a nonsingular matrix 0 ((Refer Slide Time: 18:06)) equilibrium point and 0 is asymptotically stable.

And again, several cases let me not spend time on that. So, if the Eigen values are real opposite sign, then the origin will be unstable. And so many, so many conditions. So, that is we have already learnt that. So, that is that is a special system the linear system. And now, the question is can we exploit all that knowledge to study the non-linear systems. So, that is the next question.

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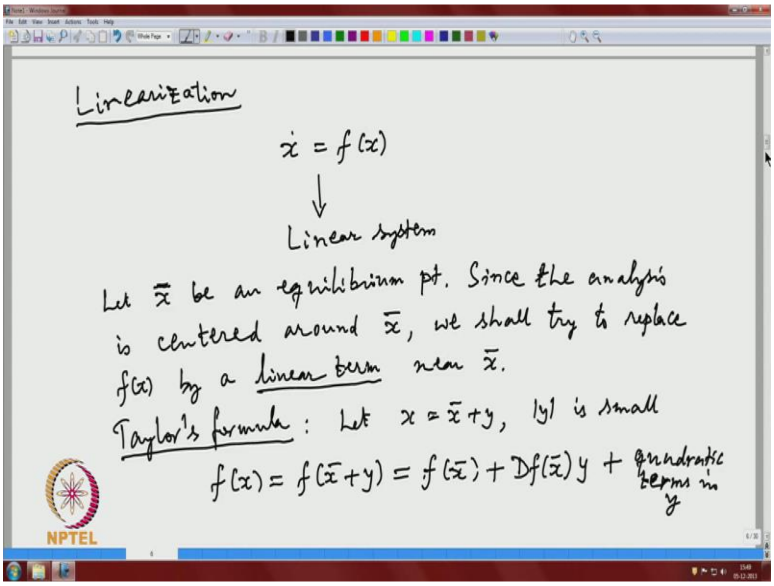

So, can we exploit whatever we learnt from the linear system and apply it to non-linear system and this is called linearization process. So, this is the first step in understanding the stability of equilibrium points of a non-linear system linearization. So, what does that mean? So, we are given this system general system our f of x. And do you want to convert this to a linear system, but where? So, again remember our interest is, so let x bar be an equilibrium, so isolated equilibrium.

And our interest is to examine whether x bar is stable or not. So, we are in a very small neighborhood around x bar. So, everything is taking place there. So, we are interested in solutions which start very close to x bar to stay there or whether they are leaving or not.

So, everything the analysis is centered around x bar. So, since the analysis is centered around x bar, we shall try to replace f x by a linear term.

So, this is linear term near x bar and again this is not new for us in calculus all the time we do it. So, given a complicated function, suppose we are interested in the nature of that function near some point. So, we have Taylor's expansion around that point. And so we can approximate that given function by a linear function or a quadratic function etcetera. So, we already have that in calculus we do that. And similarly here, so this is multivariable calculus.

So in fact, it is also called Taylor formula and it can be derived from one variable calculus and this will be included in the preliminaries. So, let me explain that Taylor's formula. So, let x equal to x bar plus y. So, y is small because, we are just concentrating near x bar. So, I would like to write that. So, f of x now will be f of x bar plus y. So, if it were just 1 variable and just 1 function, what generally we do here is x bar plus derivative of f at x bar into y plus second derivative of x bar at f at x bar y square etcetera.

So, here since we are in n dimension, n generally bigger than 1. So, that thing is replaced by what I call it D f x bar y plus quadratic terms in y. So, let me explain that. So, remember this f means there are f 1 f 2 f n. So, for each f y you can do it and then this is will be. So, this is a vector this is a vector, so this is a vector. So, y is already a vector. So, this is a matrix.

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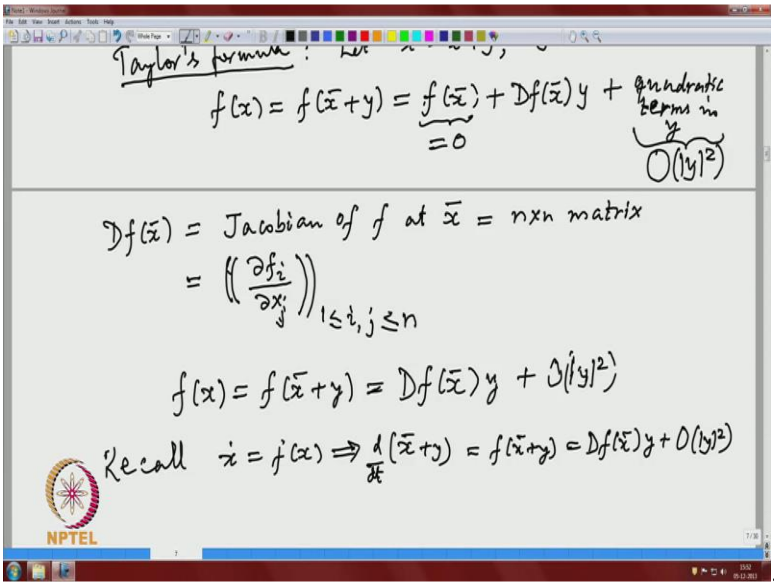

So, D f x bar is called Jacobian of f at x bar and this. So, this is a n by n matrix. So, if it is n and n matrix. So, I should tell you the entries of that thing. So, this is my matrix notation. So, I put two brackets. So, ((Refer Time: 25:00)) element is this. So, this is the matrix that is the matrix. And this one, this quadratic terms in y we write it has o of mod y square just for.

So, let me again repeat that thing, let me rewrite it. So now, we have, so f of x is equal to f of x bar plus y. Let me repeat it again not do that thing. So, x bar is an equilibrium point. So, this will be 0 plus o. So, remember now. So now, I write that. So, recall x dot now is equal to f of x. So, this implies x bar of y dot. So, let me write d by d t is equal to x bar plus y and now y, that is fine.

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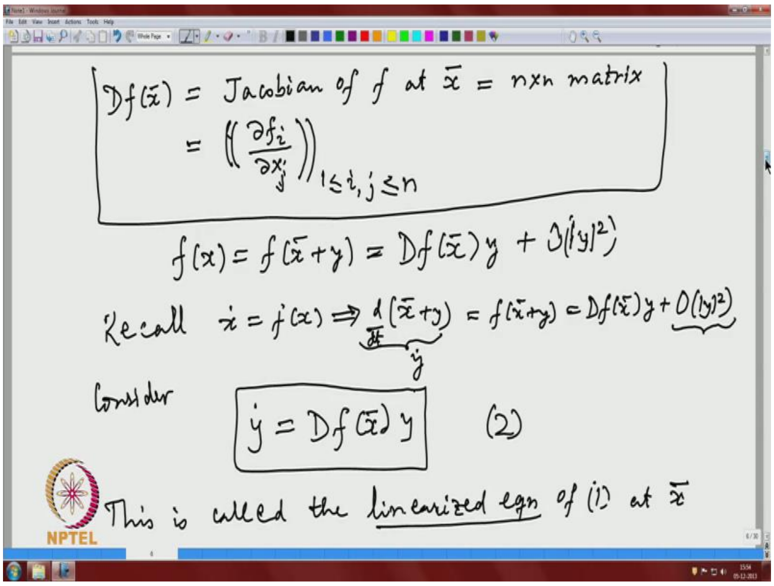

So now, we define the linear approximation. So, consider. So now, you see. So, this one this is a constant. So, this is just y dot and in the first approximation in the linear approximation I ignore this quadratic terms. So, this y dot is equal to D f x bar y this is called. So, let me call it to, this is called the linearized equation of 1 at x bar. So, that is important. So, we are just concentrating on that equilibrium point.

So, if there are more equilibrium points, then we get more linearized equations. So, each one could be different because, this matrix changes remember that this matrix. So, that changes. And one important thing I want to like to stress that. So, we started with very minimal hypothesis on the right hand side f in order to have this one.

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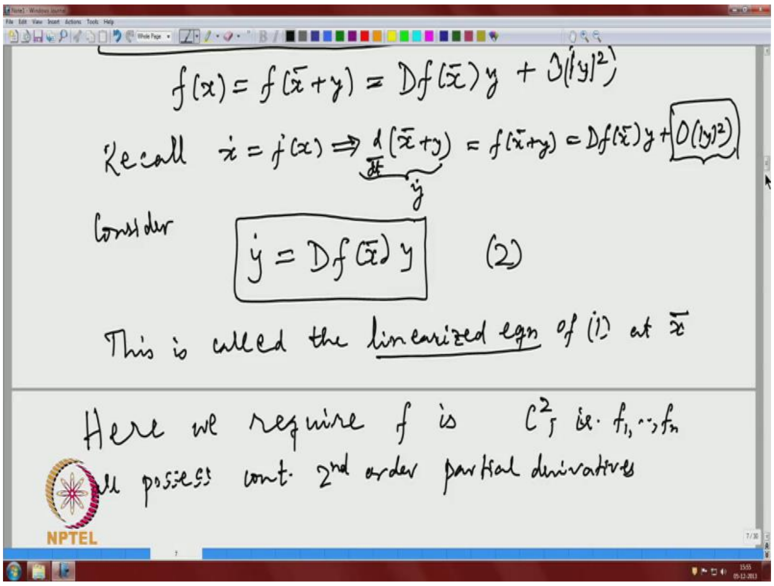

So, here we require. So, this I was telling in the beginning as we go along, we put more hypothesis on f. So, f is a c 2 function c 2 vector otherwise f is c 2. So, that is f 1 f 2 f n all possess continuous second order partial derivatives. The second order, first order already you have seen it here that is comes in the Jacobian. The second order derivatives are required in order to write this one. So, that is why we require that c 2.

But most of our examples they are all polynomials. Polynomials are in fact not only c 2, they are infinitely a many times differentiable, so no problem. So now, few more definitions we will go to examples.

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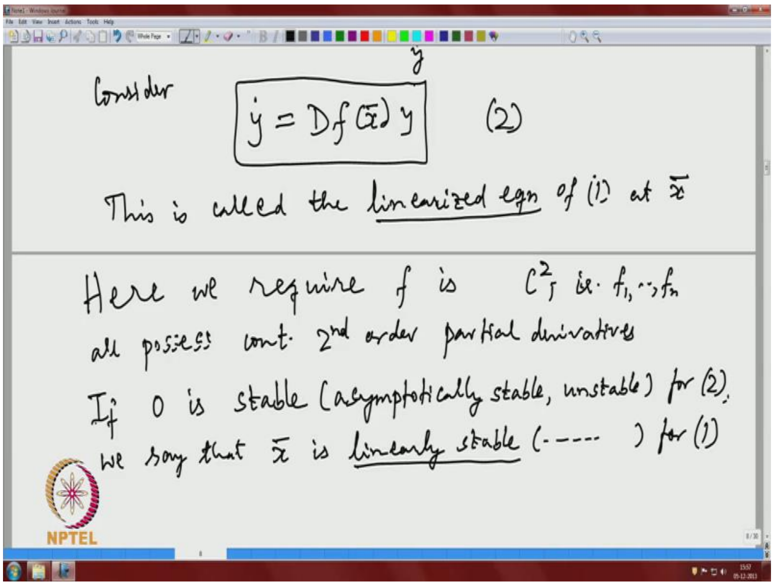

So, now this is just a linear system. So, we have to just concentrate on the Eigen values of this thing. So, let me just. So, if 0 is stable asymptotically stable this is definition stable, unstable for 2. Now, you consider the linear system 2 and decide. So, 0 is an equilibrium point for system 2. So, if 0 is stable or asymptotically stable or unstable for 2. We say that x bar is linearly stable asymptotically stable etcetera for 1.

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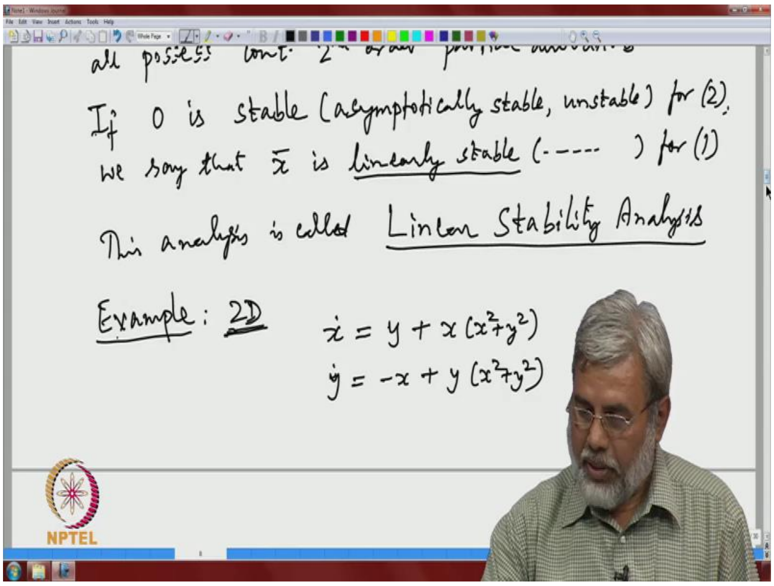

And this analysis is called linear stability analysis. So, it is not only done in case of o d's it is done in p d's and even more general equations in infinite dimensional spaces linear stability analysis. And this is routinely done by physicists and engineers. But one should not stop there. Linear stability analysis gives you something, but it may not with you see through some examples, it may not be the case with original non-linear system. So, you just consider one example.

So, example before we go on further, so 2 D. So now, hereafter the most examples will be 2 D. So, this is again a very standard one to indicate that linear stability analysis will not indicate anything regarding the original system plus x x square plus y square. there are some good cases, we will as we go along we will mention those things. So, this is a very standard 2 D example.

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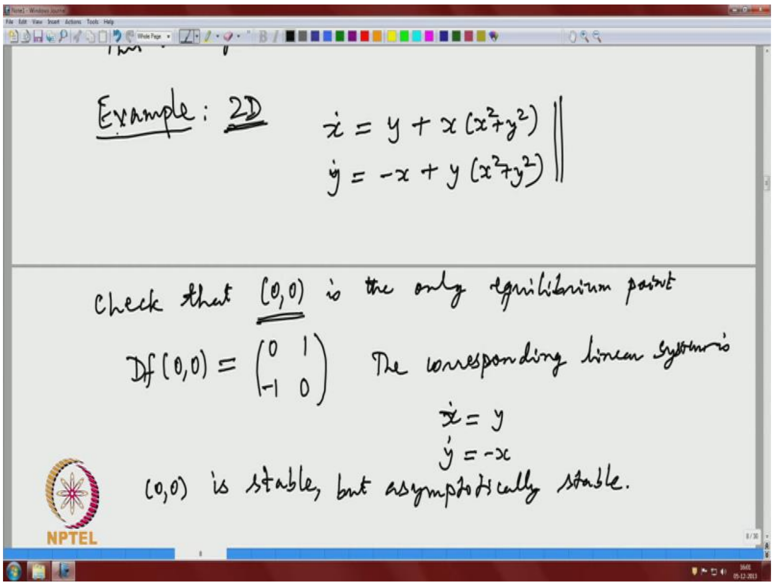

So, check that. So, sometimes it is easy sometimes you have to really do some algebra. So, this is an exercises for you, check that 0 0 is the only equilibrium point. So, let us try to do the linear analysis. So, this is my. So, I have to just evaluate the Jacobian matrix at the origin, because this is the only. So, you can easily check that, so D f 0 0. So, again this is an exercise. So, you take the partial derivatives etcetera. So, you get 0 1 minus 1 0.

So, the corresponding linear system, corresponding this is 2. Corresponding linear system is x dot equal to y and y dot equal to minus x. And we have already seen that. So, you have already seen for the linearized linear system linearized system 0 0 is stable, but not at asymptotically stable. And now, we ask the question this 0 0 is it stable for the original system? So, remember this is our original system.

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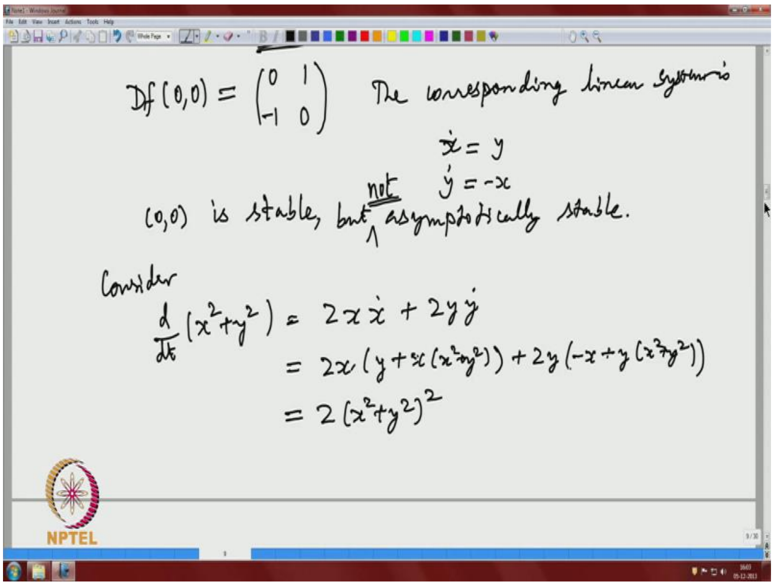

So, let us try to analyze in this case directly what will happen. So, consider d by d t of. So, this again we did in previous example also x x dot plus 2 y y dot. So, this is just y plus x x square plus y square from the original system plus 2 y minus x plus y x square plus y square. So, this is not asymptotically stable. So, we simply the algebra you get 2 x square plus y square square.

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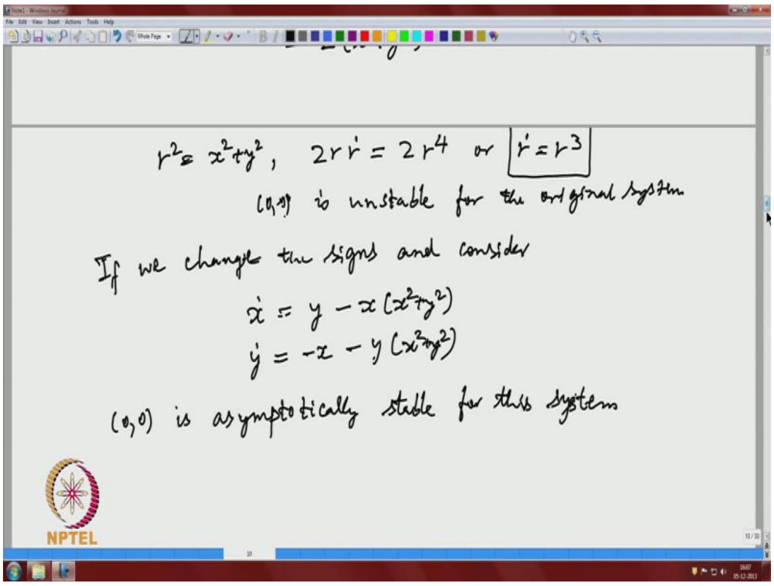

So, if I now call this as r square or just r it does not matter. So, if I put r square is equal to x square plus y square. So, the 2 r r dot is if I do this again. So, this is just r 4 or you get r dot is equal to r cube. So, for this single variable r I get a first order differential equations non-linear that is fine. And that you see this r. So, if r is positive all the time because, we are taking x square plus y square. So, r dot is r cube. So, that increases, so always increasing.

So, this the orbits the trajectories go away from the origin. So, you get 0 0 is unstable for the original system. So, the linear analysis says that this origin is stable, but not asymptotically stable. But for the original problem namely this system, the origin is unstable and if you just change the signs. So, if we change the signs. So, this is and consider this system almost same just I change the sign in the non-linear terms.

Again origin is the only equilibrium point and the linearized system does not change and in this case 0 0 is asymptotically stable for this system. So, a linear stability analysis in this example shows, that may not give the correct picture as per as the original system is concerned. And the question is it always the case?

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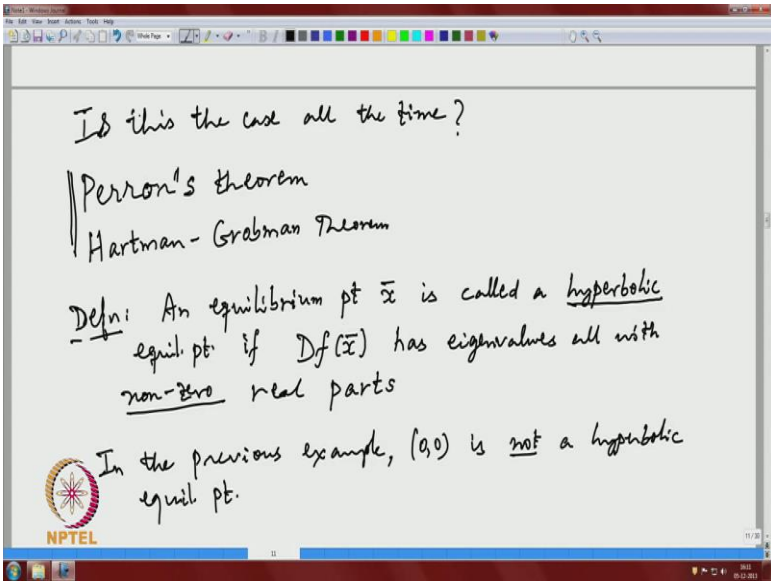

And then we cannot depend on linear stability analysis at all, if this is the case. Is this the case all the time? That is the question fortunately not. So, there are two results both in the positive directions namely, that whatever linear linearized system predicts the same is true for the non-linear system. So, these two are just mention and we will discuss may be next time. So, these are Perron’s Theorem and Hartman Grobman Theorem.

Perron’s theorem is very easy to state and even I will try to give a proof of that Hartman Grobman theorem even to state it, I have to develop certain notations. So, I will not go into that thing. But I will just mention what that theorem states. And both these concern a special type of equilibrium points. So, let me define that thing. So, let definition. An equilibrium point x bar is called a hyperbolic equilibrium point, if the Jacobian matrix has Eigen values all with nonzero real parts.

And in this case, if x bar is a hyperbolic equilibrium point. Then whatever happens to the linear system the same thing happens to the non-linear system. And that is the content of this Perron’s theorem and Hartman’s theorem, Hartman Grobman theorem. So, in the previous example 0 0 is not a hyperbolic equilibrium. And what one should do when x bar is not a hyperbolic point. And that is where Lyapunov comes into picture and Lyapunov develops a beautiful theory called Lyapunov function. And then that is essentially meant for equilibrium points which are not hyperbolic.

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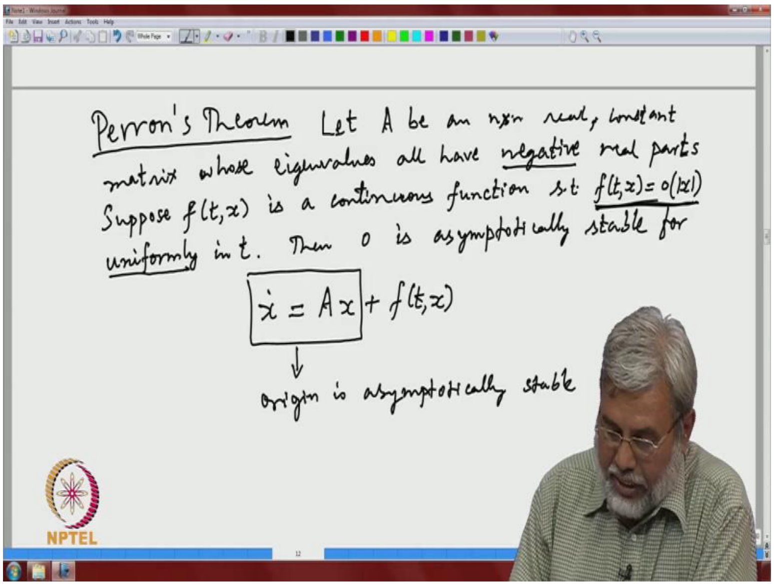

So, let me in the remaining time just try to explain what that Perron’s theorem is just statement and then. So, this is more general and it is also applicable to more situations even including infinite dimensional cases. So, so let me just try to state it. So, let a be an n by n real constant matrix, whose Eigen values all have negative real part. And in this case you know that, the linear system x dot equal to a x the origin is asymptotically stable. So, this is one of the cases.

So, suppose so in fact, this Perron's theorem considers a more general perturbations. So, let me just state it. So, f t x is a continuous function. So, we do not even worry about uniqueness. So, just take the continuous function such that. So, I will explain this a little later. So, I will write a small o of mod x uniformly in t. So, this is, so if f does not depend on t then this is fine, but this uniformly in t.

So, this relation, so this is a kind of limit relation, I will write it uniformly in t. Then 0 is asymptotically stable for x dot equal to A x plus f t x. So, this condition already implies that it is very small for x small. So, this is a small perturbation near the origin. So, if you consider this linear part and the hypothesis that the matrix, the Eigen values of the given matrix all have negative real parts.

So, here the origin is asymptotically stable. And Perron's theorem tells that. So, a small perturbation will not alter the nature of that asymptotic stability. So, even for the non-linear system it will be asymptotically stable and this is one positive result. So, let me may be next time I will give a proof of that. So, let us work out some examples the remaining time.

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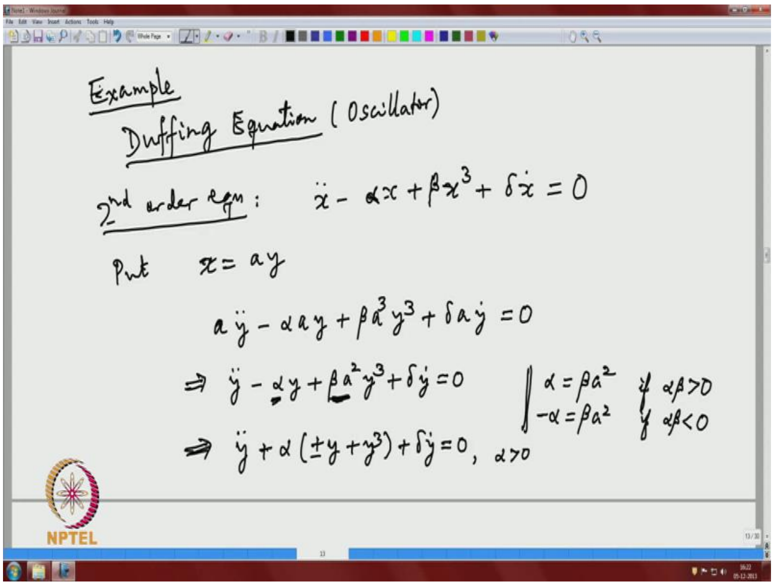

So, let me just start with one example. And try to calculate the linearized system compute this linear system and Eigen values. So, this is Duffing equation. So, you already seen this in the module giving you all the examples and this is also called

Duffing Oscillator. So, it is similar to the mass spring dashpot system. The only thing is here the forcing is non-linear, it is cubic in particular. So, let me just write that.

So, it is second order equation, so x double dot minus alpha x plus beta x cube plus delta x dot. So, let me consider the unforced one. So, I will put right hand side 0. So, we will also discuss when there is forcing term little later. So, if delta is 0 this can be put in the conservative form that we will study little later. So, when delta is positive, so this describes a dissipative system and we will see the effect of that delta positive.

So, in mathematics all the time what you do is, so there are too many parameters here alpha beta delta, is it possible to remove some of them. And so we do scaling and other things. So, let me also do here. So, that is I did not do for the logistic model. So, let me just do it for this Duffing equation. So, you will learn some scaling argument. So, put y is equal to x equal to a y. So, let me put that it is easier.

So, x equal to a y and a is a constant I am going to choose little later. What should be a? So, if you substitute this x equal to a y. So now, y is a new unknown function. So, this the first term will be a y double dot a is a constant minus alpha a y plus beta y cube plus delta a y dot. So, I am going to divide by a throughout. So, this implies y double dot minus alpha y plus beta a square y cube plus delta y dot.

Now, I choose my a. So, see here there is alpha then beta a square. So, I want to make them equal. So, I choose alpha is equal to beta a square and this I can do since a square is always positive. This I can do only if alpha and beta have the same sign. So, if alpha beta is positive and if alpha beta is negative, what I do is I get t alpha if alpha beta is. So, in that case what I have is. So, y double dot I put here plus alpha. Now, I put plus or minus y plus y cube 0 and alpha I take to be positive. So, I put plus or minus. So, that is one simplification.

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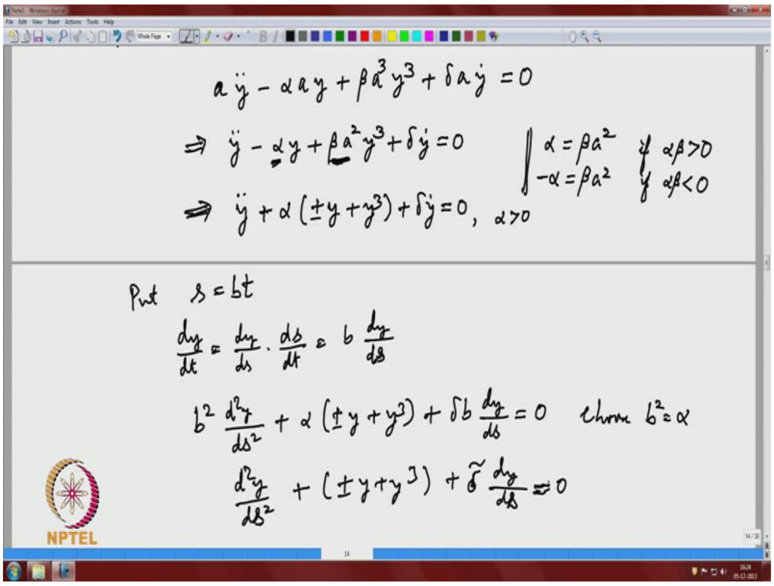

And similarly now I can change t. So, put s is equal to b t and now what you get is d y by d t is d y by d s into d s by d t and that is just b d y by d s. So, these are scales. So, I am scaling the time variable and I am scaling the unknown function x. So, if you do this thing, again what you get is? So, b square d square y by d s square plus alpha plus or minus y y cube there is no derivative here. So, that will remain same.

So just, but here I get delta b d y by d s equal to 0. And again by I divide by b or I choose alpha equal to b square, choose b square equal to alpha. And if you do that thing and then you divide by that. So, this will give s square plus plus or minus y plus y cube delta tilde that is d y by d s. So, all these exercise.

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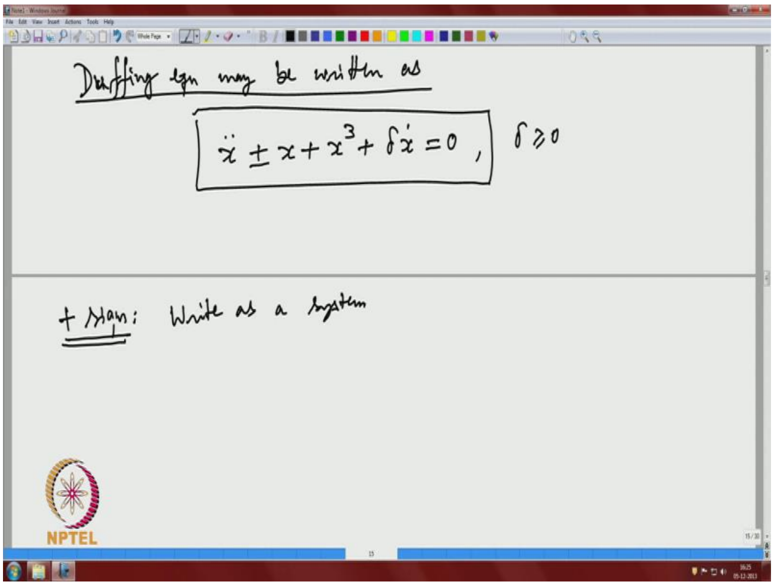

So, Duffing equation and that is what we are going to do Duffing equation may be written as. So, this is all that exercise. So, again we go back to the original variables plus or minus x plus x cube plus delta x dot equal to 0. So, you consider this simplified version. So, let me just go on little bit. So, let me just consider the plus sign. So, let us write it as a system. So, we will continue this linear stability analysis for the Duffing equation in the next class. And we will also see how the theorem of Perron is proved and more examples regarding this linear stability analysis and stability analysis in general.

Thank you.