We have been discussing about the existence and the uniqueness of solutions of initial value problems. We have, so far proved the Picard's existence and uniqueness theorem and also, Cauchy-Peano existence theorem. These proofs were involved and we used many notions from real analysis to prove the existence and a uniqueness part. If we make use of the functional analytic technique, especially, the fixed point theory, then this existence and uniqueness of the solution of an initial value problem can be very easily established without much effort. In this aspect, the Banach contraction principle can be applied to derive the existence and uniqueness of solution of an initial value problem, provided the function in the initial value problem is good, in the sense, if that satisfies lipschitz condition. Also, the Banach fixed point theorem will give us a way to compute the solution of the initial value problem. In the preliminaries, we have already seen the fixed point theorem, Banach contraction principle and also, a generalization of the Banach contraction principle.

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Let me just recall the Banach contraction principle. If x is say, x is a normed linear space and then the space is complete with respect to the norm, and we call this is a complete normed linear space. Let x be a complete normed linear space. A complete normed linear space is called a Banach space, and t is an operator from x to x, such that t is an operator; this could be a non-linear operator. So, t is called a contraction. If t x minus t y, the norm of t x minus t y is less than or equal to alpha times, norm of x minus y, for all x y in x and for some alpha strictly less than 1. Of course, this has to be a non negative number. If t is a contraction; see example; if f x is a function defined by half x, it is a linear function. So, it follows easily, that f is a contraction. If it is lipschitz continuous and the lipschitz constant is less than 1, then is called a contraction. If it is lipschitz with a lipschitz constant half, here, alpha is equal to half. Another non-linear example is f x is 1 by 4 sine x, and we have already seen that sine function is lipschitz continuous. Therefore, 1 by sine x is also lipschitz continuous with lipschitz constant 1 by 4. So, this implies that f is a contraction on r; r is a complete normed linear space, a Banach space.

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Now, the Banach fixed point theorem says or also, known as Banach contraction principle; if t from a Banach space x to x is a contraction, then t has a unique fixed point, say x star. That means, t of x star is same as x star; the point x star is a fixed point. Further, if t is a contraction, then t gives a unique fixed point. At the same time, it gives a computation algorithm to find the fixed point. Further, the sequence x n, defined by x n plus 1 is equal to, x n is equal to t of x n minus 1; so x 0 arbitrary. So, n is 1, 2, 3, etc.

The sequence x n defined by x n is equal to t x n, converges to the unique fixed point t star of t. So, this is a computation algorithm. We will see later that. This is nothing but the Picard's iterants, which we defined earlier. So, just an example of a fixed point; see, if I have function f x is equal to x square. So, obviously, x is equal to 0, that f of x is equal to 0 square, that is 0; x is equal to 0 and x is equal to 1, are fixed point of f x, is equal to x square. See, geometrically, if you look into the graph x square, and fixed point is in 1-dimensional case. The point of intersection of the function y is equal to x, and the given function. So, there are two fixed points; 0 and 1.

So, this gives you the fixed point; two fixed points are there, and f x is equal to x. All points are fixed points; f x is equal to x cube. So, you can find out the fixed point. They are the point of intersection of the graph of the curve with the diagonal line. Now, let us come back to the initial value problem.

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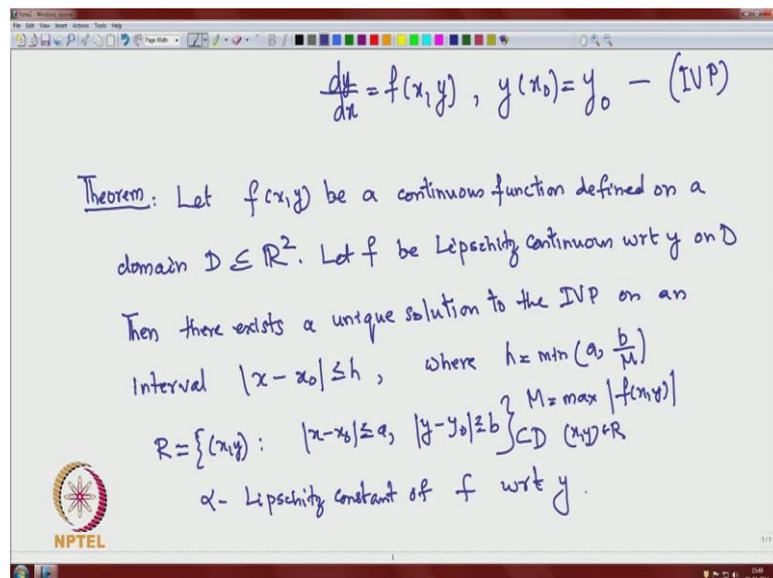
So, we have the initial value problem, d y by d x is equal to f of x y with initial condition y at x 0 is equal to y 0. Now, we state and prove the existence and uniqueness theorem for this initial value problem. The proof of the theorem will be based on Banach contraction principle. So, let me state the theorem. Theorem is, let f x y be a continuous function, defined on a domain d of r 2. Let f be lipschitz continuous on d, lipschitz continuous with respect to y on d. Then, of course, other conditions like x 0 y 0 is an integer point of t. Then, there exists a unique solution to the initial value problem. So,

this is our initial value problem on an interval, x minus x 0 less than equal to h where, h is minimum of a b by m, and m is maximum of f x y; x y is in some rectangle r, which is inside d. So, r is a rectangle, defined as earlier x y, such that x is, x minus x 0 is less than or equal to a; y minus y 0 is less than or equal to b; for some a and b, such that this r is inside the domain d.

So, the theorem is if f is continuous, it is a continuous function on d and if its lipschitz continuous with respect to y on d, then there exists a unique solution to initial value problem on an interval, x minus x 0 is less than or equal to h where, h is defined by this; where, a and b are parameters of a rectangle, which is inside the domain, and that is alpha b; the lipschitz constant of f with respect to y.

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Further, the unique solution can be computed from the successive approximation, y of n minus 1 x is equal to y 0 plus integral x 0 to x, f of t, y n of t, d t; n is equal to 0, 1, 2, etc. So, this iterative scheme, if you recall, this is known as the Picard's iterants. Same sequence of functions, which we constructed in Picard's iteration, and y 0 function is just y second, start from any arbitrary y 0 n is y 0. The proof of this theorem, we will state by using Banach contraction principle. Also, we note that in the Banach contraction principle, if t is a contraction, then t has a unique fixed point. If t n is a contraction, if t is a an operator from a Banach space x to another Banach space x, if t n is a contraction for some n greater than or equal to 1, then also, t has a unique fixed point. So, this is known

as a generalized Banach contraction principle. So, we will be using this generalized Banach contraction principle to establish the proof of existence and a uniqueness of solution of the initial value problem.

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So, let us come to the proof; proof by the basic lemma. From the basic lemma, we stated and proved earlier; basic lemma, the solvability of the initial value problem follows, if the following integral equation is solvable. The integral equation from the basic lemma y x is equal to y 0 plus integral x 0 to x, f of t, y of t d t. So, we have proved that if you want to solve the initial value problem, it is enough to solve this integral equation. You know this is known as a mild solution of the initial value problem. Therefore, we are looking for a solution of this integral equation. Let us define an operator. So, let us define first, a function; let x be a function c a b, c set of all, not a b. So, interval is we are looking for a solution from x 0 to x 0 to x 1; some for some x 1 greater than x 0; set of all continuous functions defined on the interval x 0 x 1.

It is easy to show that this x is a Banach space. If you define a norm for x in x; norm of x is supremum or maximum of x of t; t varies between x 0 x 1. It is called the sup norm or maximum. So, x is a set of all continuous functions of supremum and maximum. They are the same in this interval x 0 x 1; closed and bounded interval. Therefore, x is a normed linear space, x with this norm say, normed linear space and also, it is complete.

Every Cauchy's sequence in x with respect to this norm, converges to a limit inside. Therefore, it is a complete normed linear space; that is a Banach space with sup norm.

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$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t)) \, dt \quad \text{------} \quad (1)$$

Define an operator $T: C[x_0, x_1] \to C[x_0, x_1]$ by

$$(Ty)(x) = y_0 + \int_{x_0}^{x} f(t, y(t)) \, dt$$

If T has a fixed point, ie $y = Ty$ then the fixed pt y is a solution to the integral equation (1).

We will prove that T^n is a contraction for some sufficiently large $n \geq 1$.

Now, we define an operator. We define remember, our integral equation is y x is equal to y 0 plus integral x 0 to x, f of t, y of t d t. Now, I define an operator, define a mapping or an operator; call it t from c x 0 x 1 to itself, by t of y; it is a function at x; is equal to y 0 plus, the right hand side of your equation; call it equation 1; right hand side of equation 1; y 0 plus integral x 0 to x, f of t, y of t d t. So, if f is a non-linear function and therefore, t is a non-linear operator from the Banach space, c x, c to c. Now, if t has a fixed point; that is if you can find some y, such that y is equal to t y for some y, then that is nothing but your integral equation. Therefore, the solvability of the integral equation 1 is equivalent to the existence of a fixed point for the operator t. Therefore, my objective is to show that t has a fixed point. If t has a fixed point; that is, there exists y, such that y is equal to t y.

Then, the fixed point y is a solution to the integral equation 1. If t has two fixed points, then integral equation has two fixed points, two solutions. If t has a unique fixed point, then the integral equation has a unique fixed point. If the integral equation has a unique solution, then the initial value problem has a unique solution. So, that is an equivalence between the solution of the initial value problem and the integral equation. If t has a unique fixed point, then the integral equation 1 has a unique solution. So, we are going to

show that this operator t has a unique fixed point, and we will show that t n is a contraction for some n greater than or equal to 1. So, we will prove that t n is a contraction for some sufficiently large n greater than 1. How do we go about it?

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Let us define by definition, t of y of x is y 0 plus integral x 0 to x, f of t, y t d t; this is a definition of the operator t. Let us find what is t y 1 of x minus t y 2 of x. If y 1 and y 2 are two functions in the Banach space; let y 1 and y 2 are two points, say x 0 x 1, two continuous functions, then we take this difference and the absolute value. This is absolute value of integral x 0 to x; y 0 and y 0 will get cancelled, and what we have is f of t y 1 t minus f of t y 2 t, d t and this is less than or equal to, and also, by using the fact that f is lipschitz continuous with respect to the second argument y. Therefore, this is less than or equal to alpha times integral x 0 to x, y 1 t, minus y 2 t d t. So, this is one relation we will be calling; call it equation number 2. Now, this is also greater than or equal to alpha times integral x 0 to x. If I take a the max of or sup of sup over t, t in the interval x 0 x 1, y 1 t minus y 2 of t; you can take; this is greater than not equal to d t, and this quantity is our norm of y 1 minus y 2.

So, this is norm of sup norm y 1 minus y 2. Therefore, for this is less than equal to alpha times alpha times integral x 0 to x, norm of y 1 minus y 2 d t, which is equal to, if you integrate it. Now, norm of y 1 minus y 2 is a constant; alpha times x minus x 0 into norm of y 1 minus y 2, all right. So, if I take now, t square; my aim is to find an n, such that t n

is a contraction. Now, if this constant, alpha into x minus x 0, if I can bound this by a number which is less than 1, then t is a contraction, but this, I cannot have unless, I put some restriction on the lipschtiz constant alpha. But I do not have any restriction on the lipschtiz constant alpha. So, what I do is I take a composition; I take t square. So, t square y 1of x minus t square y 2 of x, which is nothing but the composition of t with t, two times; this is t of t y 1 of x minus t of t y 2 of x. So, this is less than or equal to, if I use 2. If I use 2, this is less than or equal to alpha times integral x 0 to x, t of y 1 of t minus t of y 2 of t d t.

Now, using the inequality; call this one as 3. If I use 3, t y 1 minus t y 2, known absolute value of t y 1 minus t y 2 from this equation number 3; I get this is less than or equal to alpha square times integral x 0 to x. So, call it t minus x 0. So, x minus x 0 becomes t minus x 0, times norm of y 1 minus y 2 d t. This, I can integrate to get alpha square by integral of t minus x 0 with respect to t, is t minus x 0 square by 2; so alpha square by 2. So, t minus that is evaluating from x 0 to x; x 0 minus x 0 is 0. Therefore, this is x minus x 0 square into norm of y 1 minus y 2. So, this is times norm of y 1 minus y 2 sup norm. Therefore, what is the conclusion? If I take the supremum of this quantity over x, varying on the interval x 0 x 1, then that becomes a norm. So, norm of t square y 1 minus t square y 2 is less than or equal to, if I take norm maximum over there, I get alpha square into x 1 minus; x 1 is the maximum; x 1 minus x 0 square by 2 factorial into y 1 minus y 2. So, I can continue this procedure.

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If I continue this procedure to take one more times t compositions, t cube y 1 minus t cube y 2, just by integrating, that integration is a process, which is making the right hand side a good number. So, t cube y 1 minus t cube y 2 which, you can show that this is less than or equal to alpha cube y x 1, alpha cube into x 1 minus x 0 cube by a 3 factorial times y 1 minus y 2. So, if you continue like this, t n y 1 minus t n y 2 can show that this is less than or equal to alpha to the power n, x 1 minus x 0 to the power n by n factorial into y 1 minus y 2. What does it say?

If n is sufficiently large, and x 1 minus x 0 is a finite quantity, and alpha is the lipschitz constant, and the denominator I have n factorial; this can be made less than 1; strictly less than 1, if n is sufficiently large. Therefore, without any additional conditions on the lipschitz constant of f, I can show that or we have seen that t n is a contraction. So, this implies that t n is a contraction, and hence, t n is a contraction for n large and hence, t has a unique fixed point by the generalized Banach contraction principle. Therefore, this Banach contraction principle, this proof gives us both existence and uniqueness. So, t n is a contraction means; t has a unique fixed point. The unique fixed point happens to be the unique solution of the integral equation that amounts to be the unique solution of the initial value problem.

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Further, the computation of the fixed point as we have seen, x n plus 1 is equal to t of x n, starting from any arbitrary x 0; that gives us a sequence x n, that converges to the

unique solution of the initial value problem, or converges to the unique fixed point. So, in our context, y of n plus 1 is equal to t of y n and y 0 arbitrary, by definition of t. So, y n plus 1 is a function of x, is equal to y 0 plus integral x 0 to x, f of t, y n t d t and it uses the initial function y 0 t y 0 x as your y 0; y 0 x is your initial condition y 0. In fact, this iterative scheme, if you compare with a Picard's existence and uniqueness theorem; this is a Picard's iterative scheme. Let us see one example. How initial condition is or how the solution is computed by using this Banach fixed point theorem, is illustrated in the following example.

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Example; let us take a very simple system, a linear d y by d x is equal to y where, y at 0 is 1, which we all know the solution. It is a linear equation. The solution y x is e to the power x; we know the solution priory. Here, what is f; f of x y is y, which is lipschitz and f is continuous with respect to both the variables, and also, it is lipschitz with respect to the second argument, and it satisfies all the properties. So, let us define the iterates. So, y 0 y 0 x is the initial function, initial condition 1, and y n plus 1 x is equal to y 0 plus integral 0 to x. Here, f n is y n, y n t d t. Therefore, what is y 1; y 1 x is equal to y 0 1 plus integral 0 to x, y 0 t d t and y 0 is your 1. So, this is equal to 1 plus integral 0 to x, 1 d t, which is equal to 1 plus x, and y 2 x is again, 1 plus integral 0 to x is y 1 t d t, which is 1 plus integral 0 to x, y 1 is 1 plus x. So, this is 1 plus t d t, which is equal to, we get 1 plus x plus x square by 2.

Similarly, if we find y 3 x, it is not difficult to show that y 3 x is 1 plus x plus x square by 2 plus x cube by 6 and so on. We can see y n x is summation, x to the power m n by n factorial, y n is x to the power m by m factorial where, m goes from 0 to n. This, as n goes to infinity, this goes to e to the power x as we expected. Therefore, the computation of the fixed point by using the Banach contraction principle; that amounts to be the Picards's. We see that this is nothing but the Picard's iteration, and by using that, one can compute iteratively, the solution of an initial value problem, provided if those good conditions of lipschitz continuity, etc. are already satisfied in the equation. So, this portion completes the existence and uniqueness of solution of initial value problem. So, we posed three problems initially; the Hadamard well-posedness problem, a model is well-posed, if a solution exists; the existence problem; and a solution is unique, the uniqueness problem; and the solution is stable, stable in the sense, if the solution changes continuously, with respect to the initial condition, then the solution is stable. All these three properties are three conditions of Hadamard.

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One is left out is the stability; the third condition. We now look into the stability aspect of the solution of the initial value problem; stability of solution with respect to initial condition. Consider the initial value problem d y by d x is equal to f of x y with initial condition y at x 0 is y 0, and if I change y 0 to say, y 0 tilde; with y 0, I have a solution say, y x and what will be the corresponding solution of y 0 tilde. What is a relationship between, if a y 0 is changed to y 0 tilde, and what happens to the solution y, and what

will be the relationship or how much close, if y 0 and y 0 tilde are very close; how close y and y tilde, the corresponding solutions; that is our question. In other words, it comes practically, in many applications, because in many experimental data, the initial data we take from the instrument from a device; while taking the reading from the instrument either, the instrument may not be that very accurate, or the person who is taking the observation is not very keenly taking; there may be a possibility of a little bit error in the initial data y 0. If a slight error in the y 0 makes a slight error, a corresponding small error in the solution, then the solution is reliable.

Then, we say the solution is stable. If a small change in the initial data results in a small change in the solution, then the solution is a stable. A small change in very small change in the initial data results in a drastic difference in the solution, then that solution is not reliable; it is not believable. Therefore, we want to make sure that the solution is a stable in this sense. So, mathematically, it is nothing but the continuity of the solution with respect to the initial error. So, stability is the continuity of solution with respect to initial data y 0. Again, if our function f is nice, if f is continuous with respect to x and y and f is lipschitz continuous with respect to y, then the stability is also guaranteed. So, I state and prove that as a theorem. If in the differential equation, f is continuous with respect to x and lipschitz continuous with respect to y, then the system, the solution is continuous, change in continuously, with respect to the initial data or the system is stabled.

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I state and prove that theorem. So, theorem is, suppose that f of x y is continuous on d, a subset of r 2, and lipschitz continuous with respect to y on d, then the solution of the initial value problem d y by d x is equal to f of x y, y at x 0 is y 0, is stable or solution changes continuously, with respect to the initial data; the initial data y 0. Proof is straight forward by using Gronwall's inequality. Proof; let y 0 and y 0 tilde be two initial data. Since, f is a lipschitz it has a solution, it has a unique solution for a every x 0, every y 0. Since, y is now with y 0, there exists a unique solution and y 0 tilde also, there is a unique solution, because of the lipschitz continuity. If we choose two initial conditions, y 0 and y 0 tilde; there are two unique solutions. Let y x and y tilde x be the corresponding solutions of IVP. So, such solution exists and such solutions are unique.

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Therefore, what we have is corresponding to y 0, we have a solution; y x is equal to y 0 plus integral x 0 to x, f of t, y t d t and with a y 0 tilde, the solution is y tilde x is equal to y 0 tilde plus integral x 0 to x, f of t, y tilde t, d t. So, there are two solutions. If I find the difference, say y x minus y tilde, and I take the absolute value, which is less than or equal to taking the difference, y 0 minus y 0 tilde, plus integral x 0 to x and since, f is lipschitz continuous with respect to the second argument, one alpha comes out. So, this is y t minus y tilde t, d t by lipschitz continuity f with respect to y. Now, by using Gronwall's inequality, we can see left hand side and right hand side; both you have y x minus y tilde. So, that can be now bounded by another function on one side. That is y x minus y tilde x is less than or equal to y 0 minus y 0 tilde into exponential alpha times

integral x 0 to x d t, and this can be bounded. This is less than or equal to y 0 minus y 0 tilde, e to the power alpha into x minus x 0 and x minus x 0; this can be bounded by h or if you take x 1.

So, that is h less than equal to y 0 minus y 0 tilde into e to the power alpha, x 1 minus x 0. What does it say? It says that y x minus y tilde x is less than or equal to y 0 minus y 0 tilde, e to the power alpha x 1 minus x 0, which is a finite quantity. Whenever, y 0; for every small change in y 0, the difference between y 0 and y 0 tilde is small, then the corresponding difference between y x and y tilde; this also is small. So, for every epsilon greater than 0, there exists a delta. The delta is given by epsilon by e to the power alpha x 1 minus x 0. If I choose this delta, then it follows that this y is continuous with respect to y 0.

Now, it is obvious that the solution is also, continuous with the initial data. Therefore, if a function, if the initial value problem, the function satisfies a lipschitz type condition, then the solution is also stable with respect to the initial data. With this, we finished the stability aspect of the solution. We have seen the existence, the uniqueness and the stability.

Bye.