Advanced matrix Theory and Linear Algebra Of Engineers
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Lecture No. # 13

Linear Independence and Subspaces

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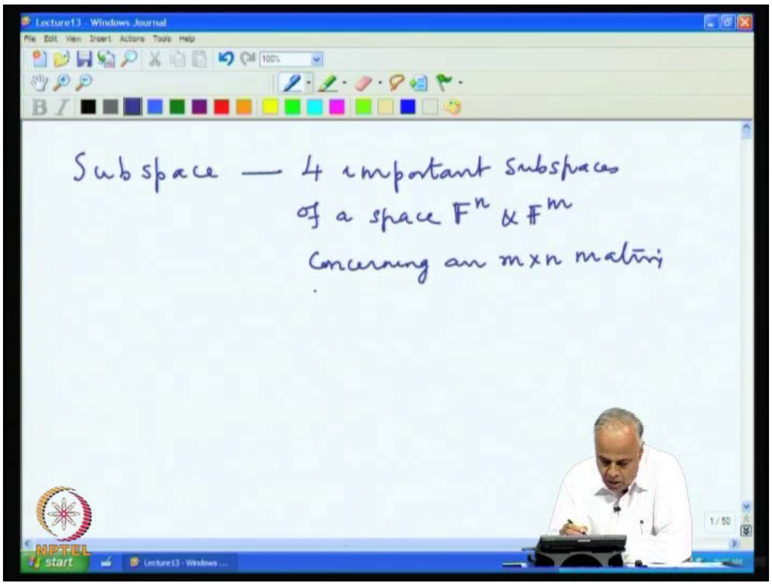

We are seeing the notions of subspace of a vector space and then, we looked at the 4 important subspace of a space F n and F m concerning an m by n matrices.

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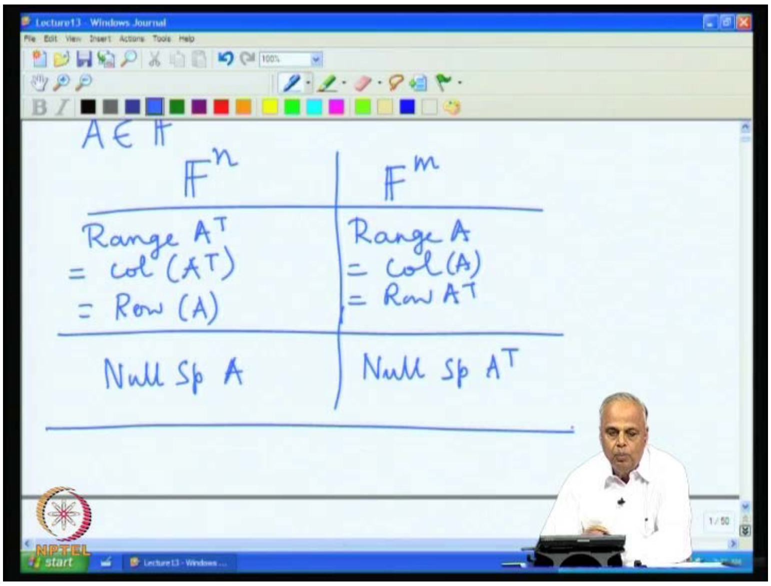

So, what are the four subspaces we had if a is F m by n matrix, then on the n component side we have two subspaces. Namely the range of A transpose, which was the same as the column space of A transpose, which is the same as the range of A the row space of the A, because the column transform or as the same as the row (a). On the other hand the F m side, we have the range of A which is the same as the column space (A) and which is the row space A transpose. So, these is the first subspace on either side and then the second subspace we have the null space of A F n side and the null space of A F m side. So, these are the four important subspace which associated with matrix a, and most of the analysis to answer our questions depends on the analysis of the four subspace. So, in order to develop the analysis we first should develop some or some techniques in the idea of the vector space.

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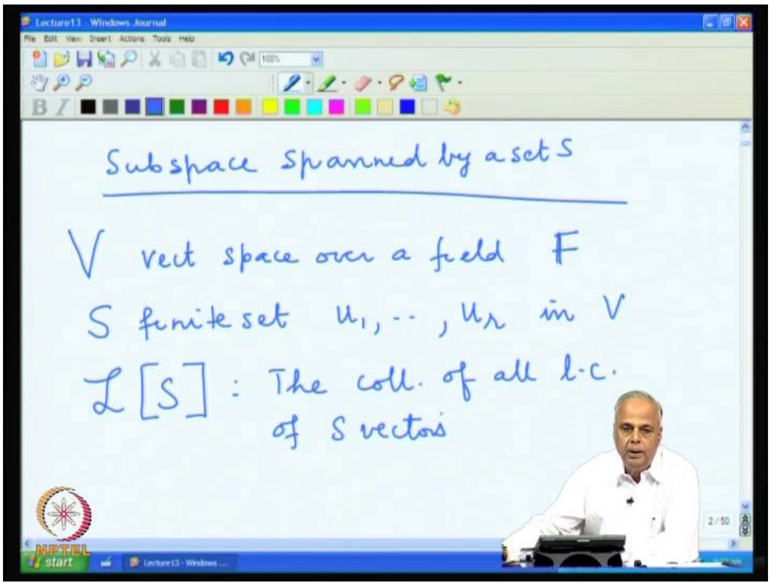

We first look out the notion of the subspace spanned by S, which already we seen. So, what we have a vector space V over the field F this is always I keep repeating always universe in (( )) the vector space over the field F. Suppose, we have a field S finite set u 1, u 2, u r in V, then we defined subspace spanned by S to be the collection of all linear combination of S vectors. The collection of all linear combination of S vectors is called as L S.

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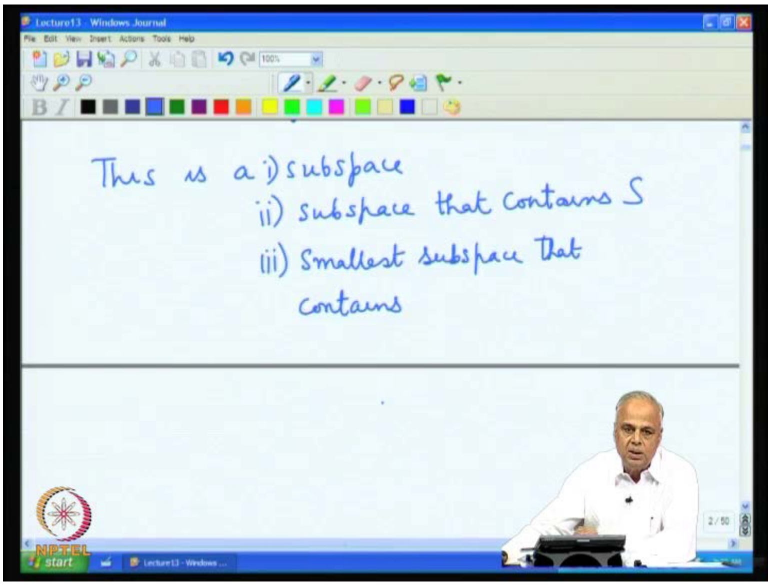

And this we have seen is a subspace first and subspace that contains S that is S is the part of the sub space and not only that among all the subspaces that contains S this is small S. So, the final clinches it is the smallest subspace that contains so, given a finite set u 1 u 2 u r the collection of linear combinations of these vectors from the smallest subspace that contains S.

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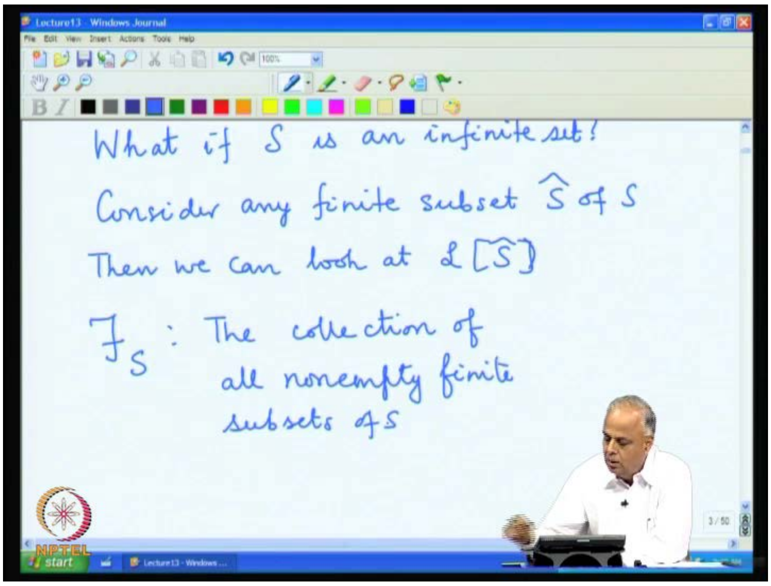

Now, what we do if S is an infinite set what if S is an infinite set that we do not know how to we take infinite linear combination, converse do not existing in the vector space. So, we avoid taking infinite linear combination of these moments and therefore, we resort only finite linear combinations. How we do this so given a set S consider any finite sub set of S lambda of S. Then we can look at L S, which is subspace spanned by S which is the sub space span by S which is the collection of all phi linear combinations set S. Because we already seen above there the moment we are given, that finite set S we can talk about L S collection of all linear combination. So, given an infinite set S we look at one particular finite set and we talk about the subspace spans by L S. Suppose, we do it for all possible finite subset of S so, let us denote by f S the collection of all nonempty finite subsets of S.

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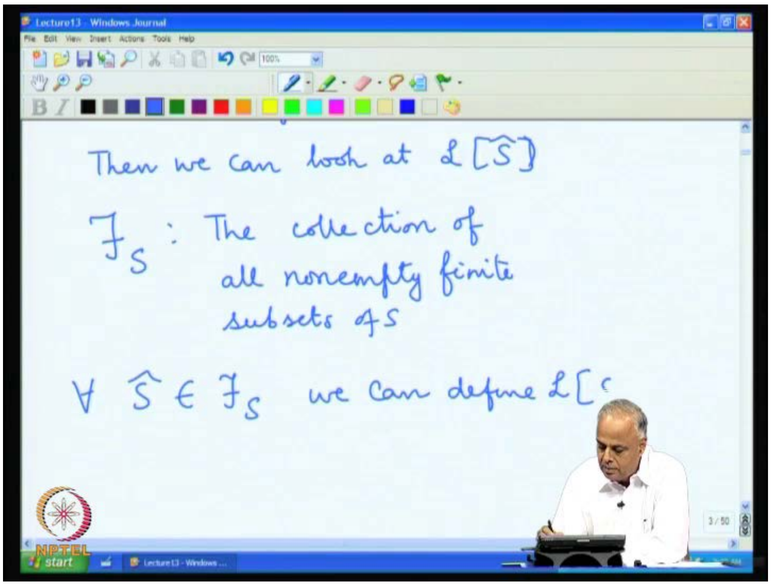

Then for every S hat f S that means for every finite subset of S we can define L S hat now, we put them all in the bracket what do we get?

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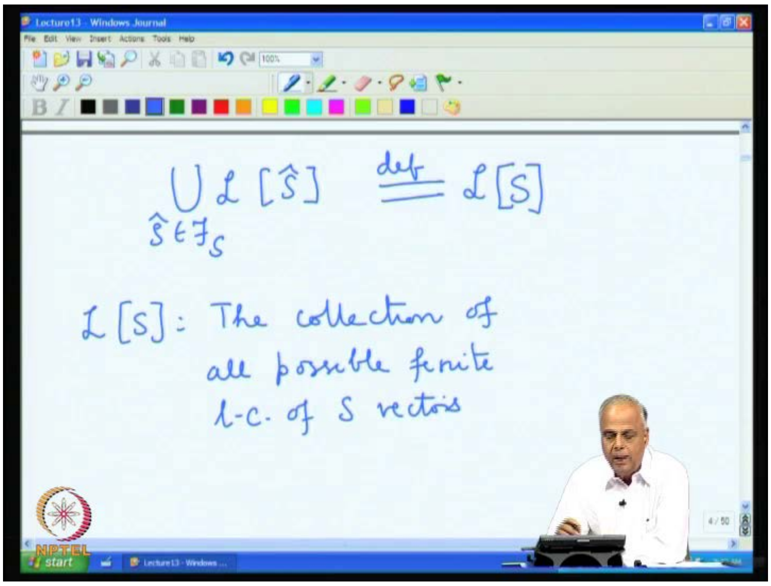

So, we get the union all the L S hats, their S set S hat is it f S, that is the collection of all subspaces spanned by all (( )) subsets of S. So, we take every possible finite subset of S and generate the L S hat correspond to this and take the union all this possibilities we get huge collection of vectors we define this to be L S. So, when S is an infinite you take all the possible finite sub subsets of S and look out the all possible linear combination of this and we get L S. So, we (( )) that L S is actually the collection of all possible since we going take only a (( )) finite so it is going to be all possible finite linear combination of S vectors, take only finite number of S vectors and look at the linear combination and do it over all possible finite set.

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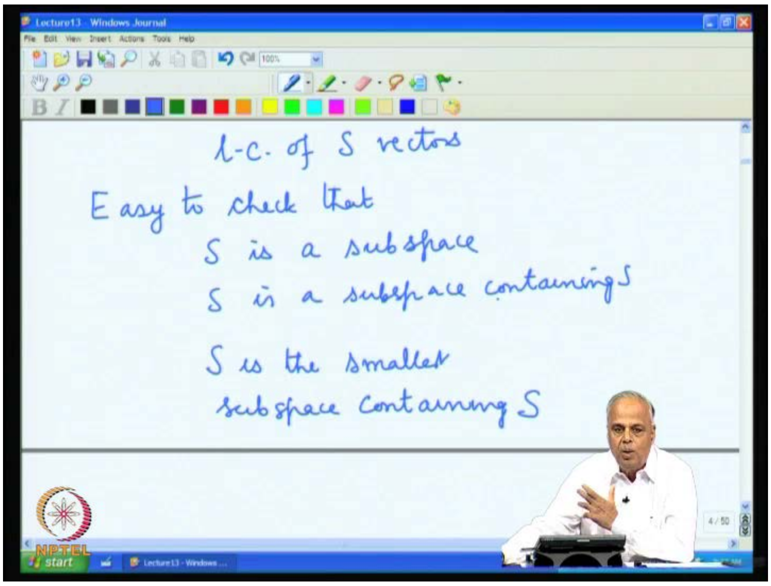

And it easy to check, that first F is a subspace and we are keep adding the adjectives S is the subspace qualifying name containing S, that is the next additional property. And then among all subspaces that contains S is smaller subspace S is the smallest subspace containing S. Thus L S in the case of infinite set has to be brought to ground level of finite set has and look at all possible finite subset and look at all the possible finite subspace expand by them you take their union and that is going to be a subspace and that subspace to the smaller subspace that contains S.

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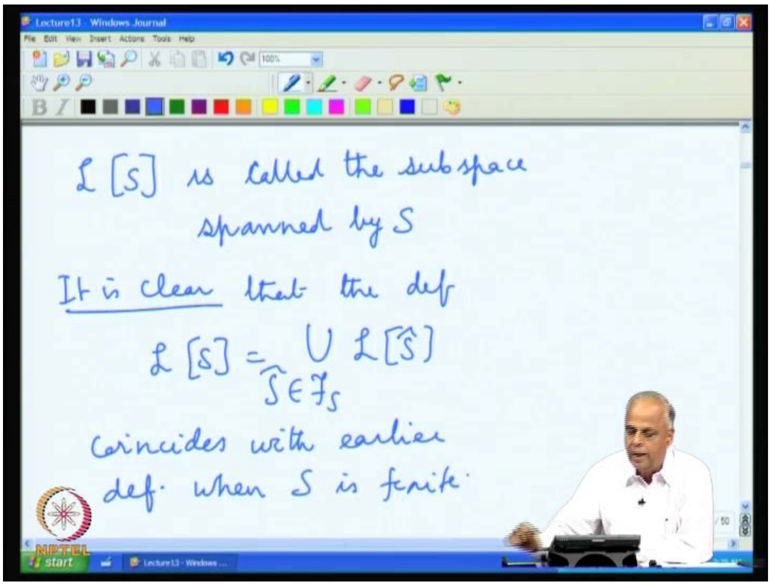

And that is called the subspace and spanned by S, L S is called the subspace spanned by S. Now, it is clear that the about the definition if the same original definition namely if we look at this infinite set were we generate all the finite sets and then you generate all the linear combination and then take all of them together give to L S. But suppose S was only a finite set and then take only finite subset and then you take only finite set obviously, S will be a part of L S and already we get L S here all other of the subsets this. So, this definition will give same L S indicate when we a finite set so and looked at the way this is proper way definition of L S for all possible sets, whether finite are infinite. So, it is clear therefore that the definition L S is equal union belong to f S L S hat coincide with earlier definition when S is finite now, we can handle the show called subspace spanned infinite set.

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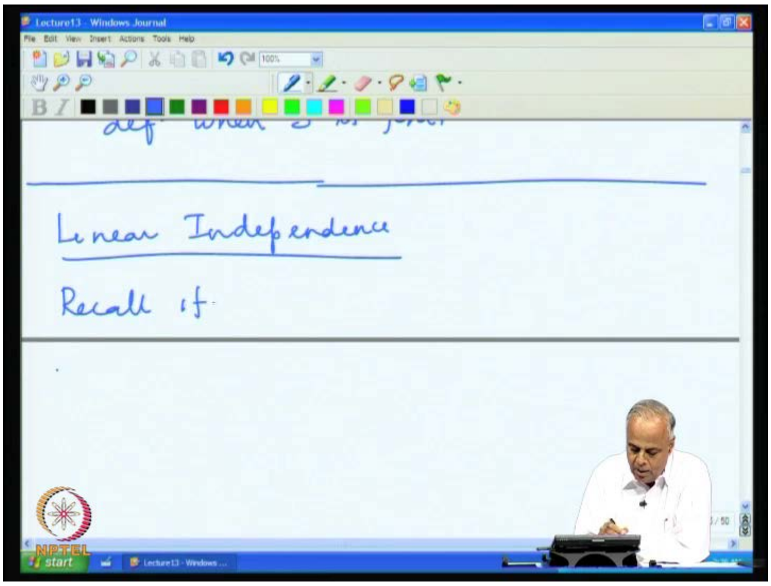

The next important thing is we had this notion of linear independence recall that notion also we had for a finite set the notion of L S finite sets from that we generated the notion of L S for an a information sets. Now, we have the notion of so far linear independent only for a finite set from that we are going to generate the notion of linearly independent for infinite set.

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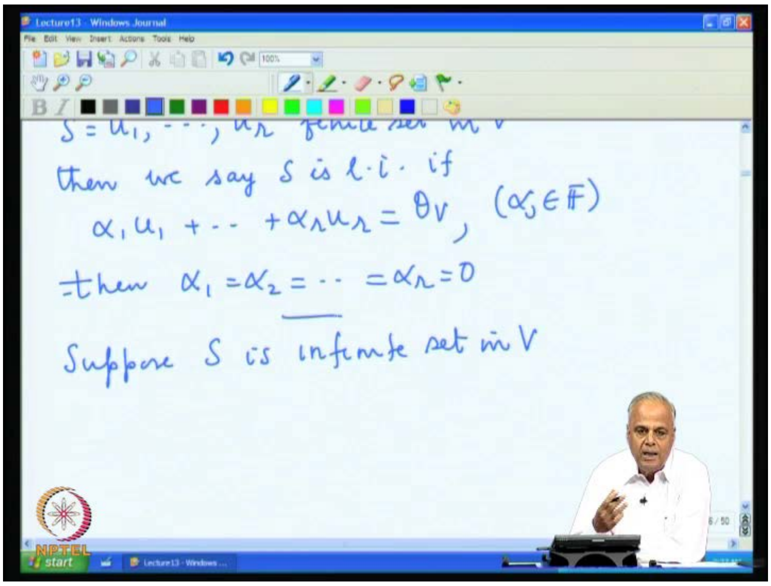

So, recall if S equal to u 1, u 2, u r is a finite set then in off course m vector space V, then we say S is a linearly independent, if any linear combination of these gives the zero vector linear combination field F (( )). If any linear combination gives the zero vector, then the only way these can happen is the all coefficient this is the definition of what is meant by linear independent of finite set of vectors. Now, we have and a infinite set how do we define so, suppose S is infinite set in the vectors space V, then what we do as before (( )) we picking up the finite sets of these. So, every finite subset of this linearly independent then we says S is linearly independent.

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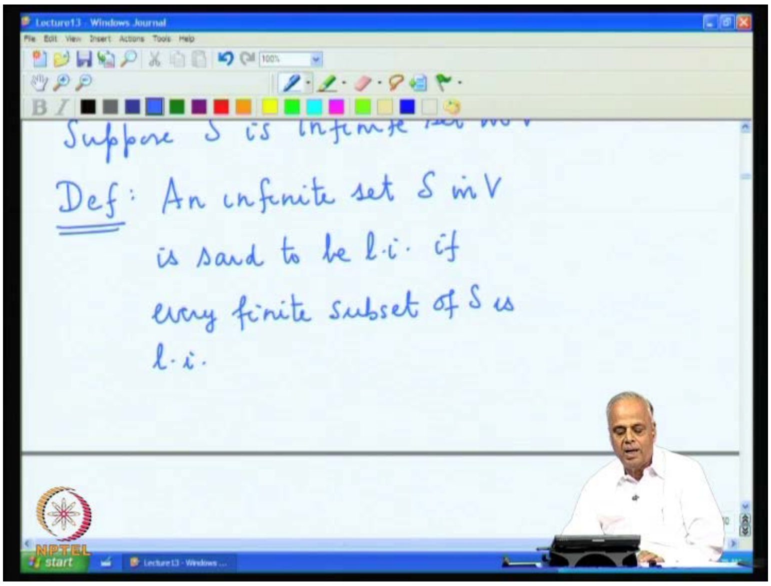

So, definition an infinite sub set in V is set to be linearly independent, if every finite is linearly independent set sub set of S, the moment we have the finite sub set of S so we can whether it is in so every finite sub set of S is linearly independent, then we say the set S is linearly independent.

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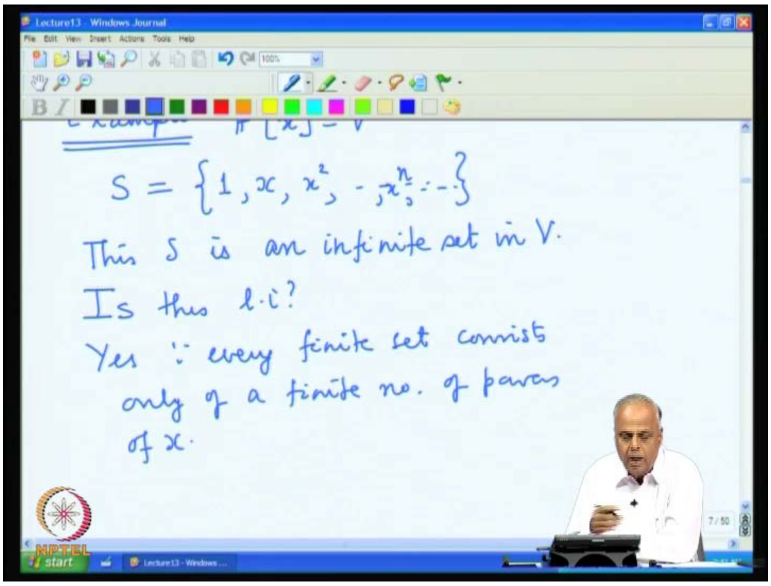

Let us look at the simple example let us look at the set F the vector space to be space of all polynomials over the field F, that is the set of all the polynomials with the coefficient field F. In these set up let us consider S to be the set of vector in V what are the vectors in V vectors are in polynomials in V in this S the vectors in V are polynomial the coefficient (( )). So, we are going to consider S to be a collection of polynomials the polynomials that we are consider 1 x, x, square and so on these is an infinite set. Take all possible polynomials with powers of x running through the nature of numbers. So x to the power of 1 x to the power of 2 so on and so forth we get infinite number of polynomials, which means we get infinite number of polynomials in the vector space V, because vectors in the space are all polynomials.

So, let us consider the infinite set so, this S is an infinite set in V is the linearly independent, to check whether it is linearly independent we are to make sure every finite (( )) linearly independent. So, what will be a finite sub set it consist of certain numbers of powers S. Now, if we take a certain number of powers of x look at the linear set to be polynomial in the polynomial is zero only co efficient of zero, because the zero the one with all the polynomials are zero. And hence, the answer is hence, because of every finite set consists only of a finite number of powers of x and hence, as I said as a combination of x polynomial and that will be zero different only as the coefficient of zero. So, therefore, every finite subset is linearly independent and hence the original set is linearly independent.

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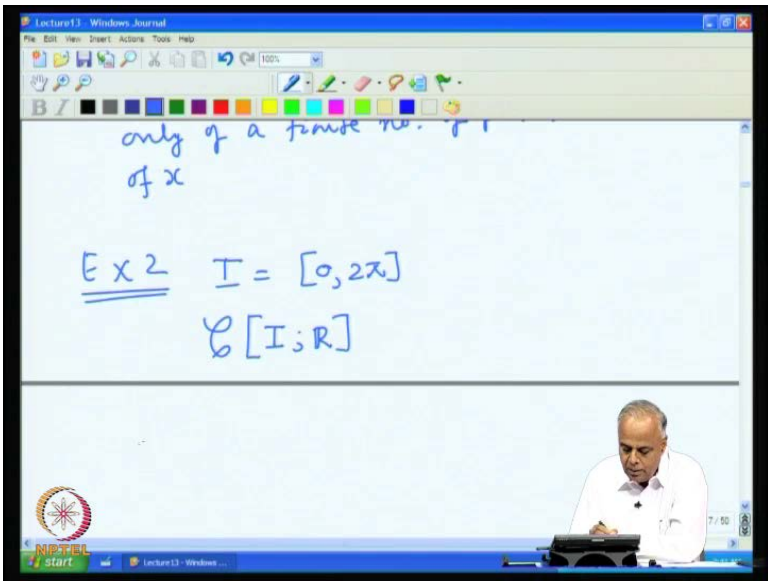

Let us look at another example, take the interval zero to 2 pie and consider all set of continues function from I to R.

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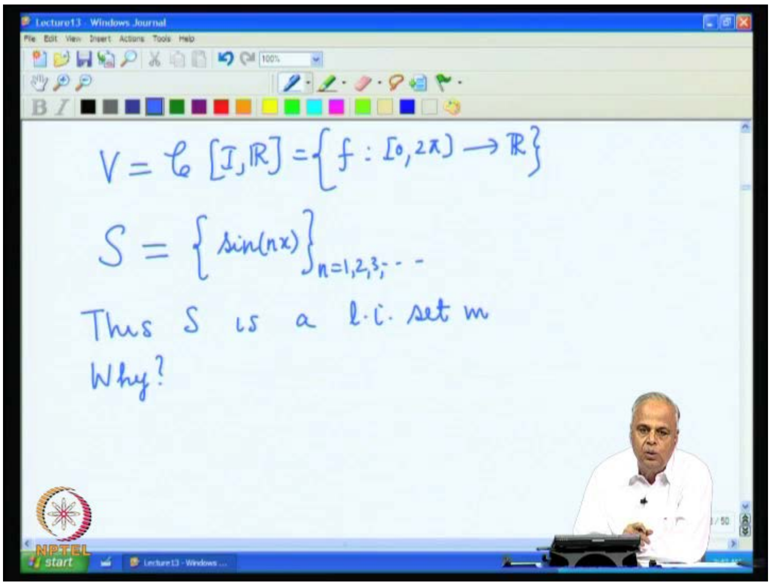

That is C I R recalled the definition is the collection of all functions, which match the functions in the interval zero 2 phi and take the real values so, is the collection of all real value functions defined on the interval zero 2 pie. On this vector space consider the set of all these functions sin n x, where n is 1, 2, 3 extra previously look at x to the power of n and run through the integers. Now, look at the sin n x runs through the integers this S is a linearly independent set in the vector space V, where our vector space is the correction of all continues functions define on the interval zero to pie taking real values. Why what is that to show S is linearly independent to show finite set is linearly independent.

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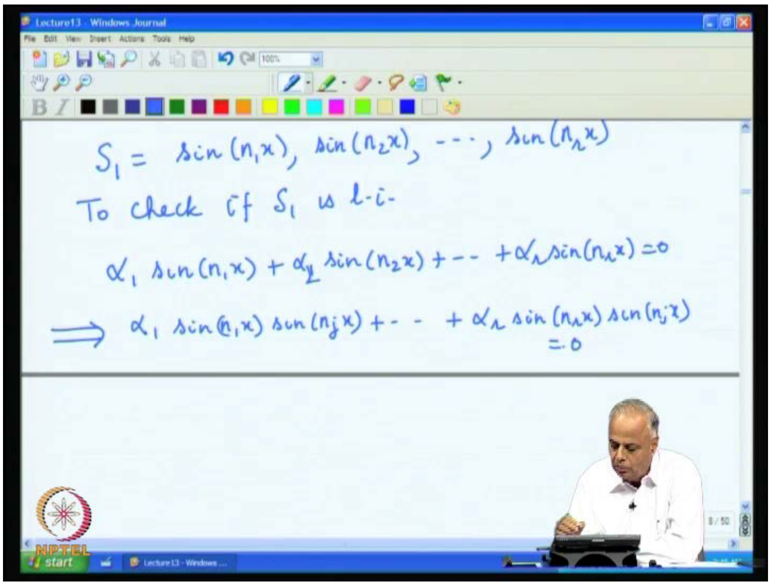

Consider any finite subset now, what do you mean by a finite subset of S consists of all n s. So, we considering the sum of n s for finite numbers of value n S 1 consist of sin n 1 x sin n 2 x and so on sin n r x. So, we have a finite subset of S, where n 1 n 2 n r are integers so, take any finite subset, if you show that linearly independent and hence you get S is linearly independent. So, to check if S 1 is linearly independent (( )) take an arbitrary finite set and verify it whether it is linearly independent. So, what do we have to check whether S 1 is linearly independent so, we start with linear combination alpha 1 sin n 1 x plus alpha 2 sin n 2 x and so on alpha r sin n r x equal to 0, zero means the function which take the value zero at all the point x in the interval zero to 2.

So, to check S 1 is linearly independent, we look at the linear combination like these and see if all the co efficient are 0. Now, these is true we can multiply by sin n j x and integrate, so we will get sin n 1 x in to sin n j x plus etcetera plus alpha r sin n r x in to sin n j x equal to 0.

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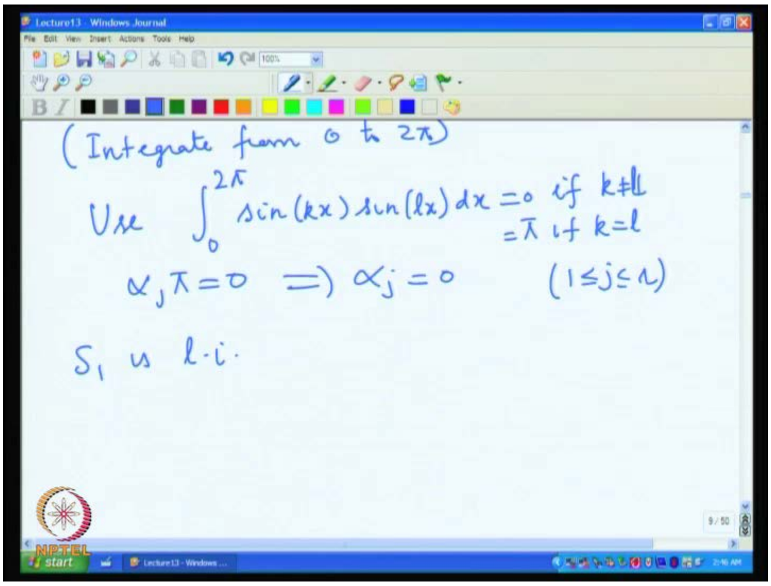

Now, integrate from 0 to 2 phi using the property of sin we see that the first term is 0, the second term is 0. Because use the fact that integral zero to 2 phi sin k x into sin l x d x when k l are integers is 0, if k is not equal to l and phi 1 is k equal to 1 if using the property. In these integral every term in vanish until we hit n j, where will get sin squared n j x and that will be phi and so will get alpha j phi equal to 0, but since phi is not zero that will say alpha j equal to 0. So, for every j if we do this from 1 lees than equal to j less than equal to n, we get all the co efficient will be zero therefore, S 1 is linearly independent, since S 1 is arbitrary finite subset of S is also linearly independent.

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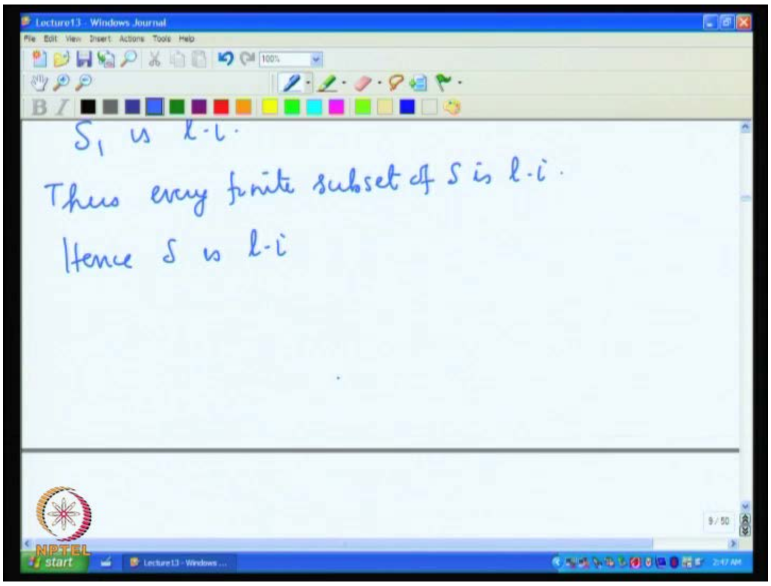

Thus every finite subset of S is linearly independent hence S is a linearly independent, say thus motion of linearly independent on finite set comes that every subset must be finite, subset of S 1 is linearly independent.

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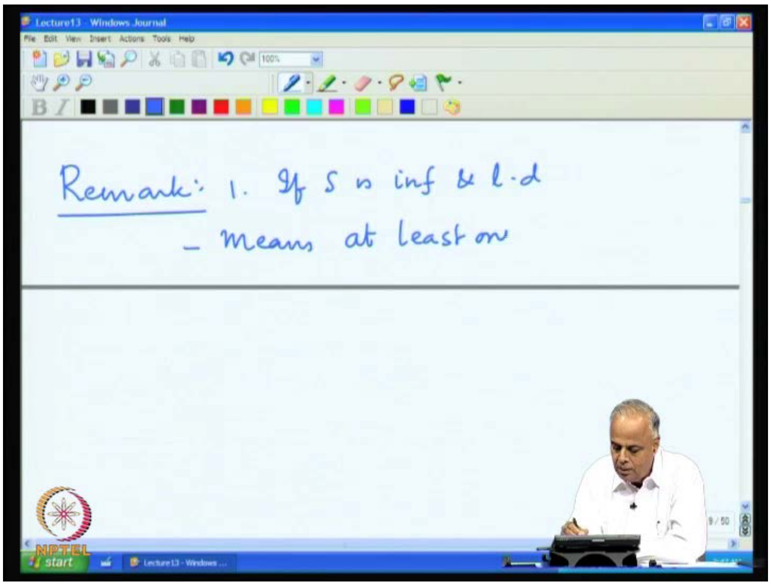

So, immediate remark you must note is that, if S is infinite and linearly dependent what is mean by linearly dependent, if not linearly independent it must be linearly dependent. Not linearly independent means what? Negation of linear independents, linear independents means every finite subset is linearly independent, negation means at least one finite subset will not be linearly independent, that means at least one subset.

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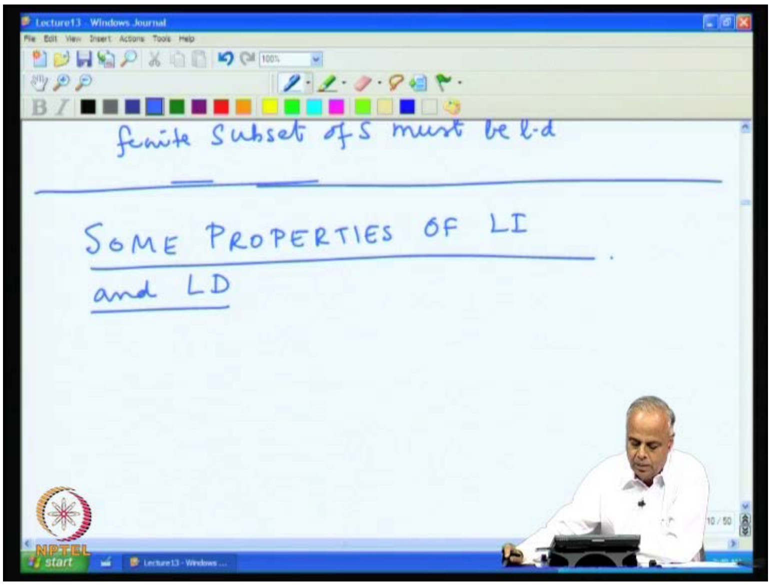

What finite subset at least one finite subset of S must be linearly independent, that is the important property of linearly independent and linear dependent, when you are talking about in finite subset. So, now we have an extended the notion of the sub space find by S to even infinite sets, we extended the notion of linear independent and linear dependent two sets, which are also infinite. So, therefore, we do not have worry about finite and the infinite sets, but it makes the lot of difference, when the sets that we are going to deal with finite or infinite. Now, we look at the some properties of linear independence and linear dependence. So, look at some very useful important property, which will come in and the very often in our analysis.

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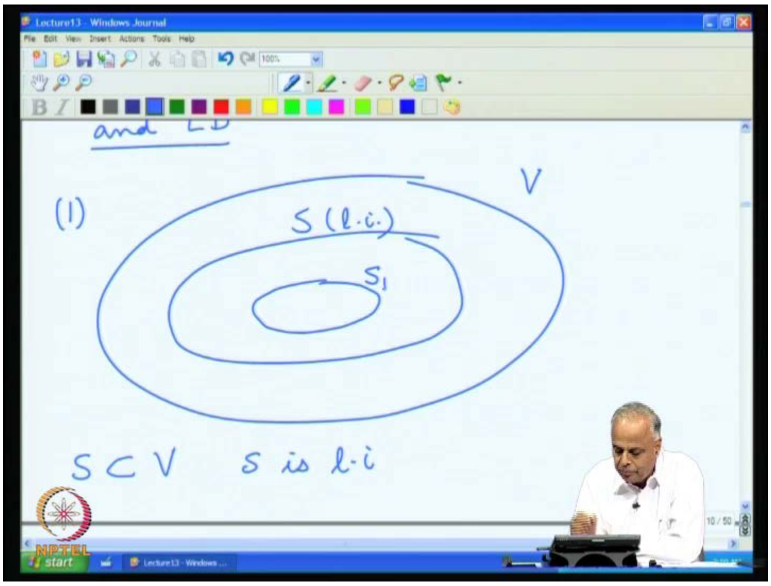

The first property so, we have a vector space V suppose we have subset S, which is linearly independent. So, suppose S is contained in V and S is linearly independent. Again, we repeat that in the back ground always a vector space V over F that is our universe, where all these discussion takes place is in vector space over a field F. Suppose, a vector space V and I have a subset S, which is linearly independent so these S is linearly independent. Now, look at a non empty subset of S.

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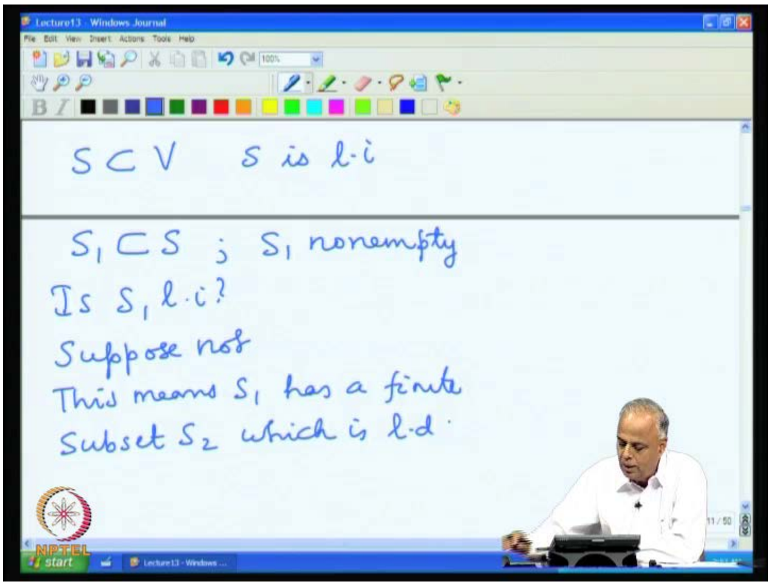

So, we have S 1 contained in S, S 1 is non empty what can we say about S, is S linearly independent, is S 1 linearly independent. So I have a linearly independent set inside, that I am taking the subset will the subset also have the same property of linearly independent. Suppose, not what does that means, this means this set S 1 is not linearly independent, the moment S 1 is not linearly independent, it must have a finite subset which is linearly dependent. So, this means S 1 has a finite subset S 2, which is linearly dependent.

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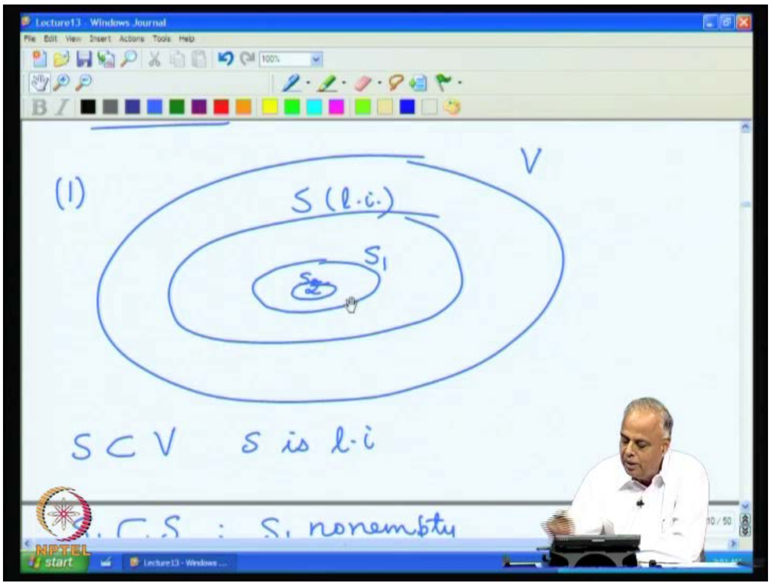

So, get back to the picture, we had linearly independent set S and inside that sitting as subset S 1, the subset S 1 is such that inside that is the S 2, which is linearly dependent. Now, S 2 is sitting inside S 1 and therefore, it is also sitting in S therefore, S 2 is finite subset of S and therefore, S 2 is a finite subset of S, which is linearly dependent.

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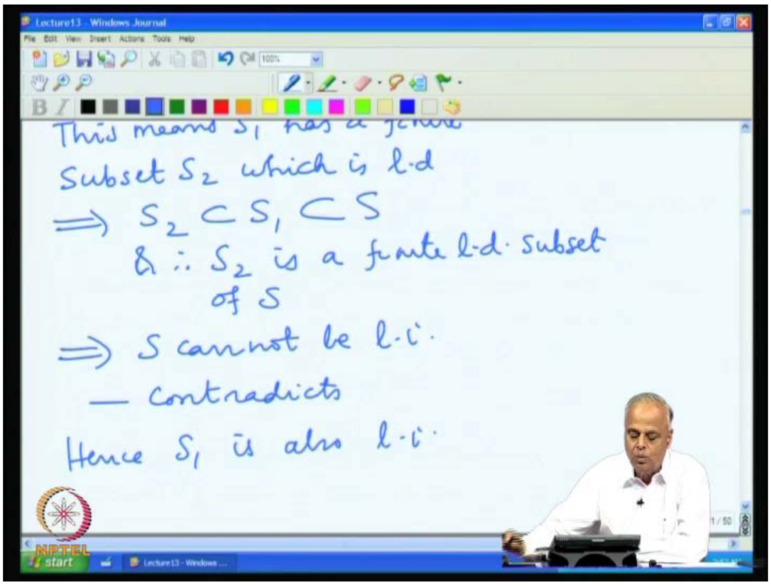

So, what this gives is that this implies (that this implies) S 2, which is contained in S 1, contained S. And therefore, S 2 is a finite linearly dependent subset of S. Now, this S cannot be linearly independent, because S here linearly independent every finite subset must be linearly independent, but here we have a subset which is linearly dependent and hence S cannot be linearly independent. This contradicts starting point that, we started with linearly independent set yes we started with linearly independent set yes. And assuming that S 1 is not linearly independent, became to a wrong conclusion that original set was not (( )). Hence S 1 is also linearly independent.

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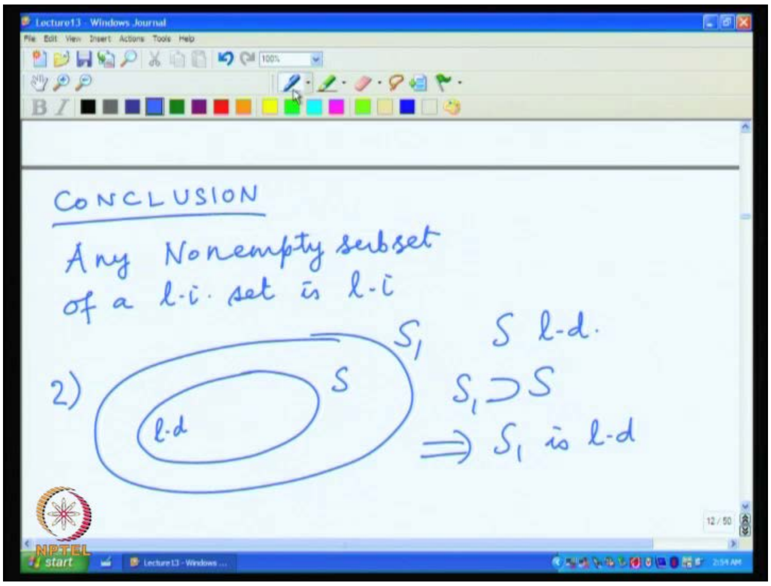

So, what is the conclusion? The conclusion the first important property of linearly independent set is any non empty subset of a linearly independent set, is linearly independent. Subsets are linearly independent sets like a (( )) property, because sitting inside the linearly independent, it possess the automatically that g of being linearly independent. And analog this property is that we start with set S, which is linearly dependent, anything above that must be linearly dependent. So, S linearly dependent S 1 is a super set of S implies S 1 is linearly independent, because if S 1 are not (if S 1 are not) linearly dependent it will be linearly independent, if it where linearly dependent anything setting inside must be linearly dependent. Therefore, S must be linearly independent, which we contradicts the fact that we started the linearly dependent and hence any super set of a linearly dependent set.

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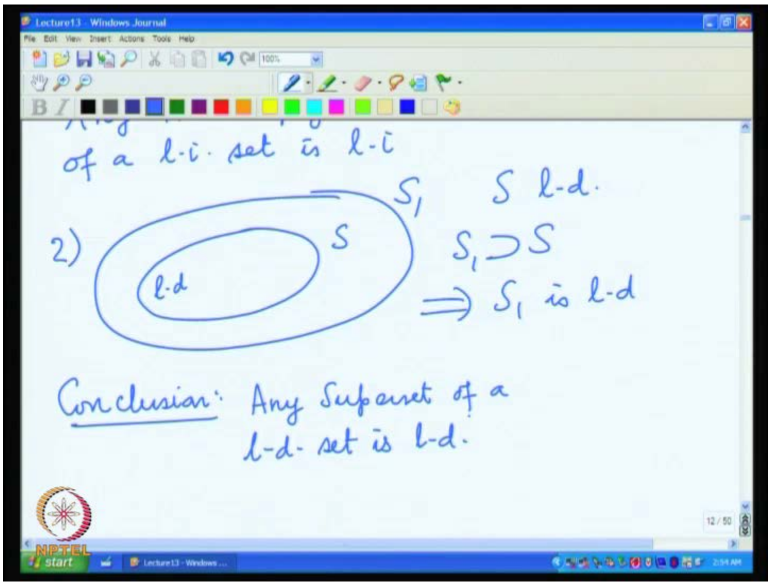

So, conclusion is that any super set of a linearly dependent set is linear dependent that is a analog dual property of previous fact that any subset of the linearly independent.

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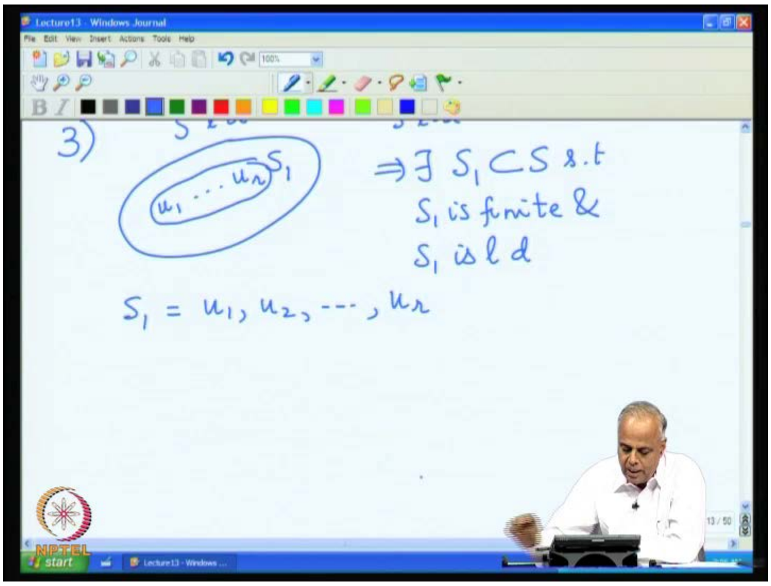

Now, let us look at the linearly dependent suppose, I have a linearly dependence set as what as does act means, linearly dependent means, there will be a finite subset, which will be linearly dependent, because if every subset was linearly independent then the set S would have become linearly independent. So, S linearly dependent implies their existing set S 1 contains S such that, S 1 is finite and S 1 is linearly dependent. So every linearly dependent must possess a finite linearly dependent subset. So, here is this is the set S 1 u 1 u 2 u r, so I have set linearly dependent the moment it is linearly dependent, it contain a finite subset S 1 which is linearly dependent. So, we are writing S 1 u 1 u 2 u r.

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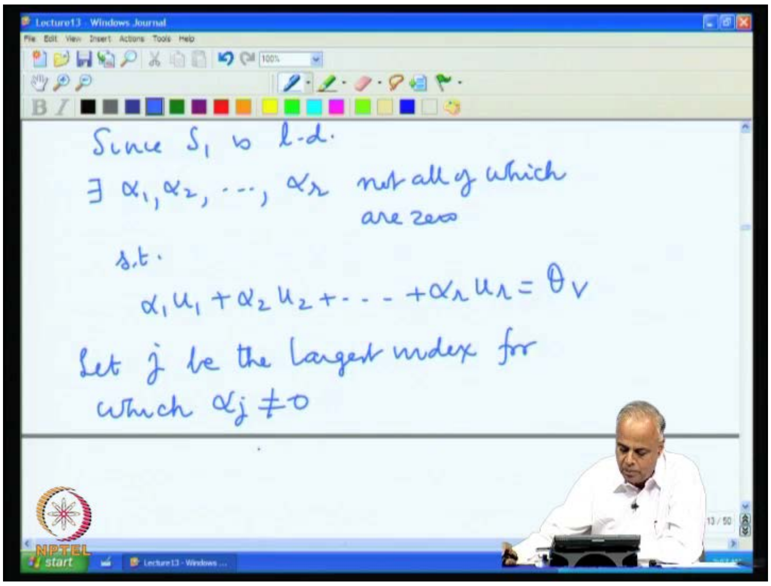

Now, since S 1 is linearly dependent what does that means, a set is linearly dependent, finite set is linearly dependent. If you can write a combination linear combination, which gives zero vector without all the coefficient being zero. So, linearly dependent there exist alpha 1 alpha 2 alpha r not all of which are zero, such that alpha 1 u 1 plus alpha 2 u 2 plus alpha r u r will be zero vector. Now, we look at this alpha 1 alpha 2 alpha r, all them are not zero, so (we keep going) we keep going see, which is last non zero alpha, it may be alpha r or may be alpha r zero, but alpha r minus 1 is not zero or it may be that alpha r is zero alpha r minus 1 is zero, but alpha r minus 2 is zero. So, we go and find which is the last index, for which alpha is not zero.

So, let alpha j be the largest index for which alpha j not equal to zero, you know that all of them not equal to zero. So, keep going until behind there nothing is available everything is zero. So, j is such that alpha j largest, the longest among them, behind with nothing else is not zero and everything will be zero, this means alpha j pulse must be 0. Because j is the largest index for which alpha j is 0.

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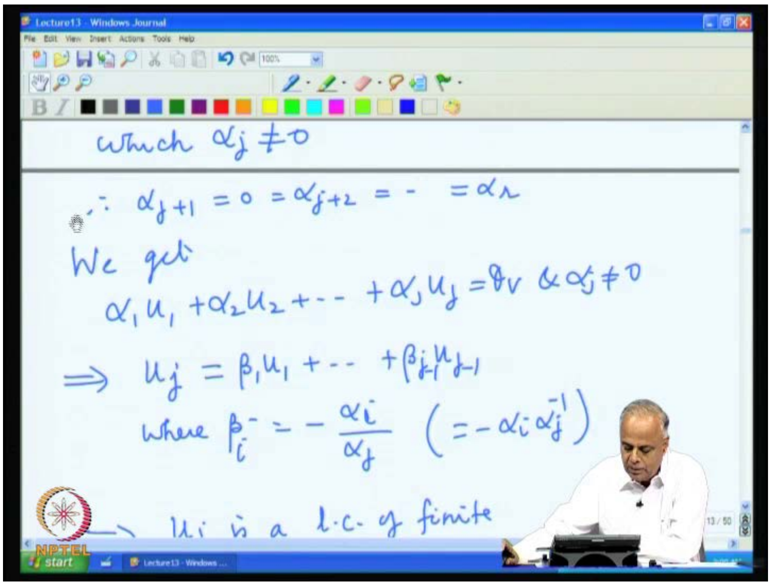

So therefore, alpha j plus 1 is 0 alpha j plus 2 must be zero and all way up to alpha r is zero, because j is r than everything is not alpha r is itself is not 0. So, now we have what does that means, look at this starting point suppose, u 1 u 2 are a linear independent. Then must have the combination like this since, behind j alpha j are zero, this sum will terminate at alpha j u j. So, we get alpha 1 u 1 plus alpha 2 u 2 plus alpha j u j is theta v and alpha j is not 0. That is the case, we can write u j an equal to beta 1 u 1 plus etcetera beta j u j, where beta j minus alpha I put it beta I is minus alpha i by alpha j it should be beta j minus 1, you take u j as 1 side and all the others to other side. And this is possible, because alpha j is not zero, in general division means multiplying by alpha j inverse; this is actually minus alpha i into alpha j inverse the (( )). This means, u j is a linear combination of finite number of vectors in S.

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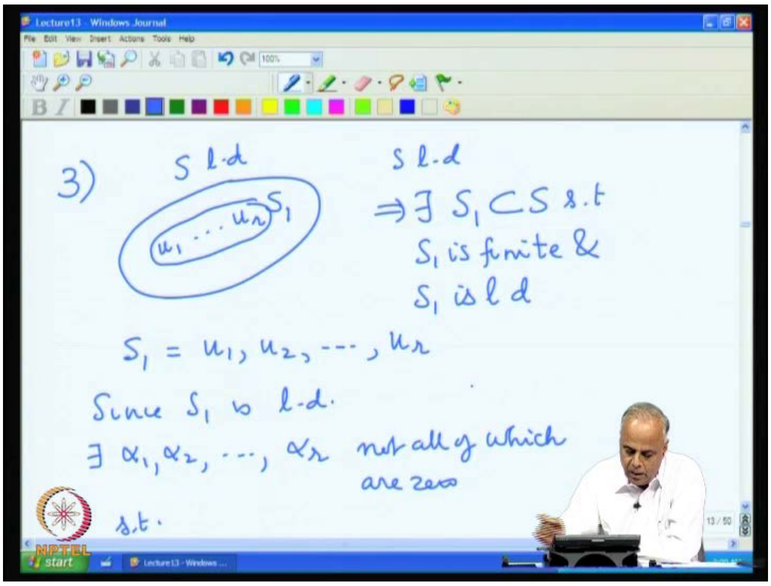

So, let go through it what we have done, we are taken circle S and we are so linearly dependent. So, we have linearly dependent set and therefore, we got a finite subset and therefore, we get a vector it is a linear combination of finite number of vectors. So, the conclusion is, if you start with linear dependent set there must be at least one vector, which is linear combination of finite number of vectors in this, there will be least one vector which is linear combination of finite number.

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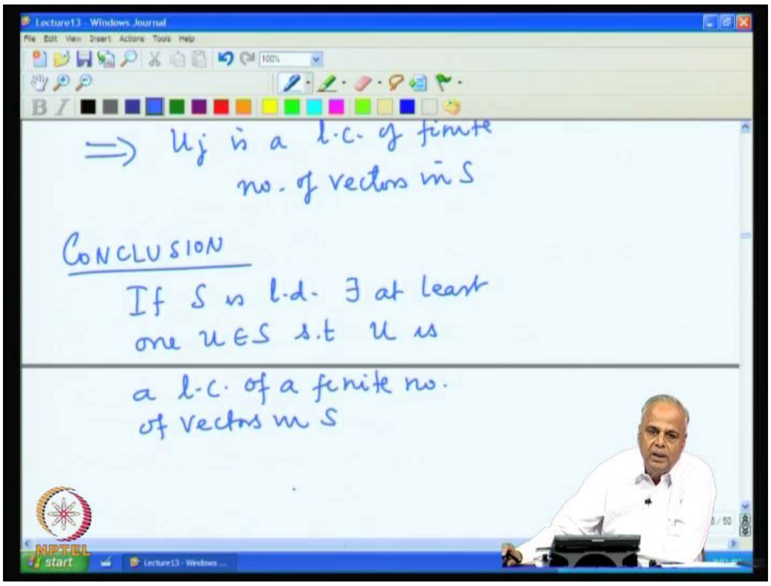

So, the conclusion of this discussion is that, if S is linearly dependent there exist at least one vector u (( )) S, such that u is a linear combination of a finite number of vectors in S. This is the third important property of this linear independent and dependent, the first 1 was every non empty subset of linear independent (( )). The second one was dual result that every super set of linearly dependent set is independent set. The third is whenever, we have a linearly dependence set at least 1 vector is the finite linear combination of the others.

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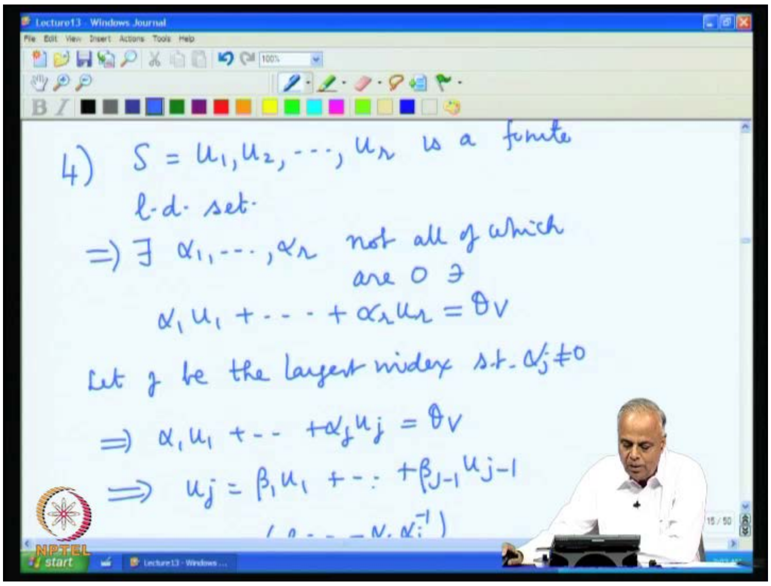

Now, let us look at the specialization, when we look at of finite set so suppose, S equal to u 1 u 2 u r is a finite linearly dependent set. Now, so linearly dependence set, we must have a linear combination, which is zero. So, third coefficient being zero like, what we have just now done. So, there exist alpha 1 alpha 2 alpha r not of all of which are 0 such that alpha 1 u 1 plus alpha r u r is equal to theta v once again, look the largest index for which alpha j is not 0. So let j be the largest index, such that alpha j is not 0 therefore, as before we get alpha 1 u 1 plus alpha j u j equal to theta v again, as before we get u j is beta 1 u 1 plus etcetera beta j minus 1 u j minus 1, where beta i is minus alpha i alpha j minus 1 this calculation exactly same as did it before.

Now, what we get it is we have the set of vectors u 1 u 2 u r less, we are arranging them in that orders. So, finite set of vectors, which are linearly dependent which are arrange in the sequence u 1 u 2 u r, If you do that, than there is the vector u j, which is the linear combination of all the previous follows u 1 u 2 u 3 u j minus 1. So, there is the vector u j, which is the linear combination of all the preceding vectors.

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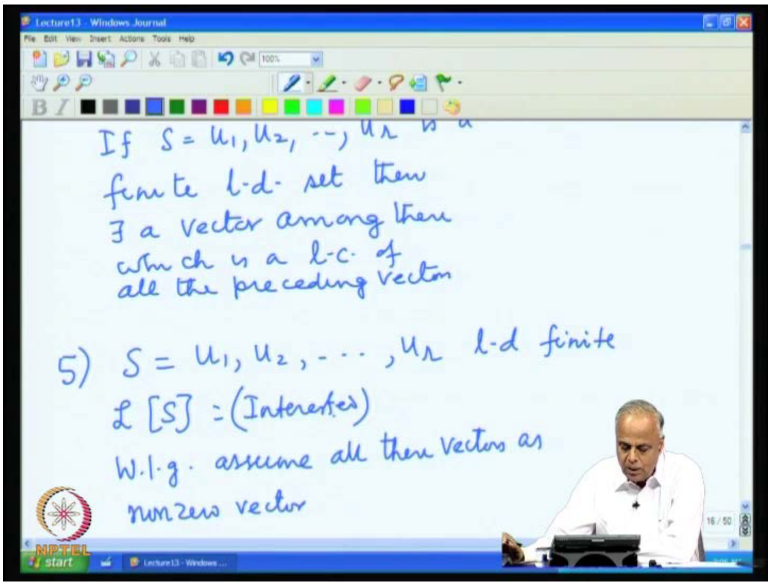

So, what is the conclusion? The conclusion is that, if S equal to u 1 u 2 u r is a finite linearly dependent set than there exist a vector among them, which is linear combination of preceding vectors, if you have a linearly dependent set finite number of arrange the first second third fourth and so on somewhere, along the line, where we a vector which is the linear combination of all earlier vectors. So, that another important property, we use simple facts there often. The next we keep expand this idea further suppose, we have again a finite linearly independent, I am sorry linearly dependent set. Suppose linearly dependent and finite. Now, what does our analysis just now says, it says that it 1 of the them must be a linear combination so we are interested in this (( )).

Now, if any of the vector u 1 u 2 u r is the zero vectors is no way going to help the building blocks, because whatever, multiple of zero we get take you are going to zero. And whatever, vector is add to zero vector we are going to the same vector. S0, zero is not going to add anything your building process, without loss of generality, if there are any zero vector we can through them out. So, without loss of generality assume all these vectors as non zero vectors. So, we have finite set of vectors u 1 u 2 u r, they are non zero and there are linearly dependent, where is the finite set of vector, which is linearly dependent. Now, our analysis just now says whenever, your such a set, there must be a vector, which is the linear combination of all the preceding fellows, there must be vector among these which is a linear combination of preceding vectors.

Now, we do not know which one it is well it may be that, you find is a linear combination of previous vectors u 22 is linear combination of previous vectors, u 13 is a linearly combination of previous vector. So, have many vectors we know that there is one there may be many vectors, which is linearly combination of the previous that could be 1 that could be 2 that could be 25 vectors, which are linear combination of previous. We do not know all assured by the previous analysis, if that at least one vector which is linear combination of the preceding vectors.

So, now what it is we look at this set S that first explain what will do, let us look at this set S start moving from the left, can u 1 be a linear combination of preceding fellows no way, because u 1 is the non zero vectors there nothing previous where cannot be expressed (( )). So, u 1 is not a linear combination of the proceeding vectors, it may be u 2 is a linear combination or maybe not. So, what we do, I go from the left the moment I het vector, which is linear combination of previous ones I stop, that means I look at the first vector from the left, who is a linear combination of previous fellows. Suppose, I have a vector hear, which is a linear combination of previous fellows then I strike it out, let me L S where more clear.

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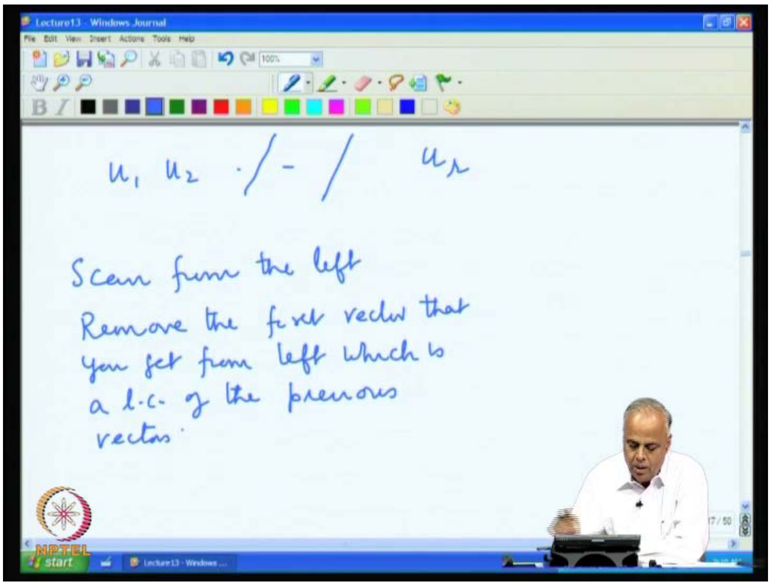

So, I have u 1 u 2 and so on u r, when I go for the from the left, I keep searching for vector which is linear combination, I see u 2 are linear combination of u 1 no then I go to u 3 and so on. When I hit the first vector, which is a linear combination of previous vector, I delete that fellow. I just remove him from that set, then what I was r minus 1 vector, which is the subset of the original set S. Because one of the vectors from S has been removed, then again after removing whatever, is remaining I continue the process again. Now, again search from left to right now nothing, before this can be linear combination of previous fellows, because this was the first vector that came about the linear combination of previous vectors.

So, the next, if at all any other vector it will came later suppose, we hate another vector, which is a linear combination. Now, all the previous vectors with this removed is I know them remove. So, this remove it this fellow is linear combination o the previous, then we remove that, when we continuous this process in the finite set it is eventually, ultimate end giving a finite subset, which is now linearly independent, because nothing is the combination of the previous vectors. So, how do you this so therefore, scan from left remove the first vector that you get from left which is a linear combination of the previous vectors.

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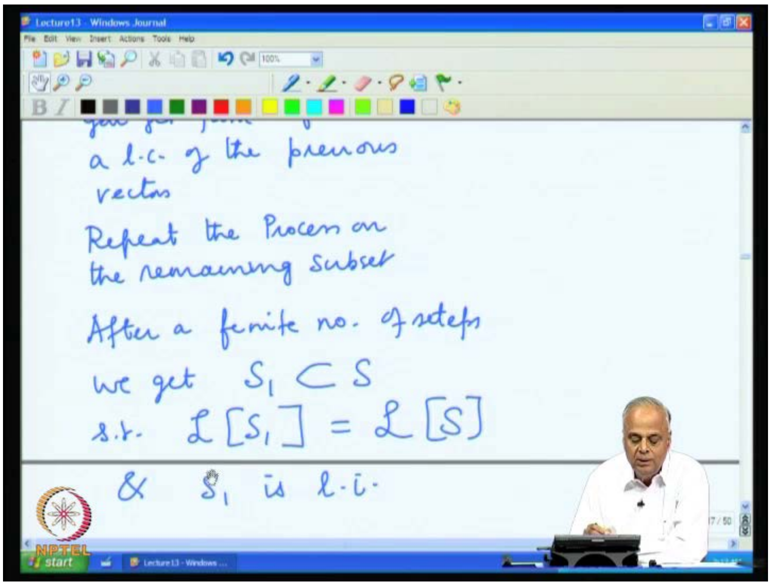

Repeat the process repeat the process on the remaining subsets after the finite number of set, because we are in finite set this has to terminate, this process has terminate, because we keep moving to right and there is only that much space is right (( )). So, they cannot move beyond it is the stage remove one vector, totally only r vectors it will be removed more than r vectors therefore, after a finite number of steps. We get a set S 1 which is strictly contain is S such that. Now, whatever was removed by the linear combination of remaining fellows, so therefore, they are not going to produced anything new in the spanning L S 1 and S 1 is linearly independent. So, therefore, the conclusion is that we have a finite set S, which is linearly dependent then the subspace spanned by S can also this spanned by the subset S 1, which is linearly independent.

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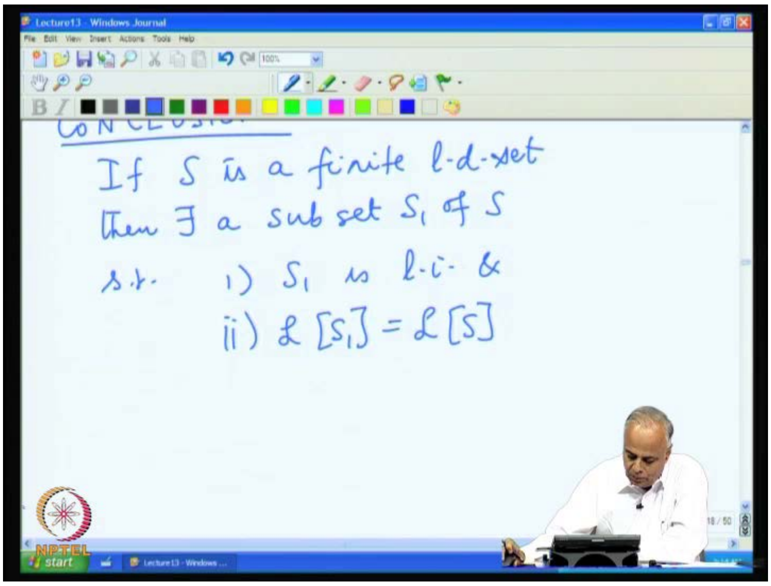

So, now let us summaries as conclusion, if S is a finite linear dependent set, then there exist a subset S 1 of S such that, one S 1 is linearly independent and S 1 also spanned same as S. What we are doing is a linear dependent set as the lot (( )). Systematically, we are trying to remove all the redundant information and retain only the relevant pure information, which comes under linearly independent subset S 1, given any linearly dependent set S, there finite linear dependent set S there exist the finite linearly independent subset of S such that it spans same space has been original values. We shall look at one more property which was number 5.

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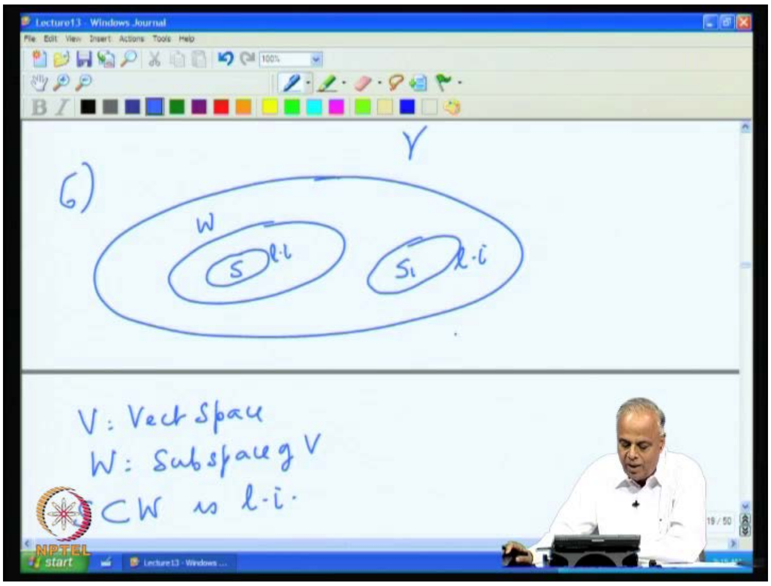

So, next one is something, which will use again suppose, we have a vector space V and we have subspace W and inside the subspace, we have linearly independent set, this is linearly independent. So, we have V vector space, W subspace of V and S containing W is linearly independent. So, we are consider sub space V inside, which we have a linearly independent set. Now, suppose, W is small part of V either lot of space left outside W, in these space suppose, I pick up a set S 1 which is linearly independent. So, S 1 is contained in V, but not contained in W is sitting outside. So, S1 in V not in W and S 1 is linearly independent. Now, it is not difficult to verify that S union to S 1 will also to be linearly independent in V.

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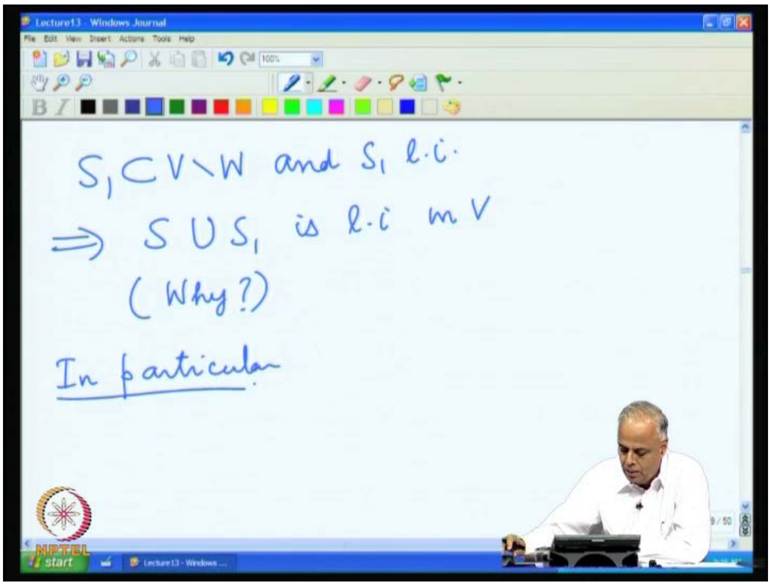

This implies S union S 1 is linearly independent we shell leave the verification exercise, so simply write why? You just verify, it is simple process suppose, it is not linearly independent and then you get the conversion. So, if you take a subspace and inside, the subspace, you take a linearly dependent set. And pick up the linearly independent set outside so you take a linearly independent set inside W. And we take linearly independent, set outside W where to together will be a independent set in the V, so in particular.

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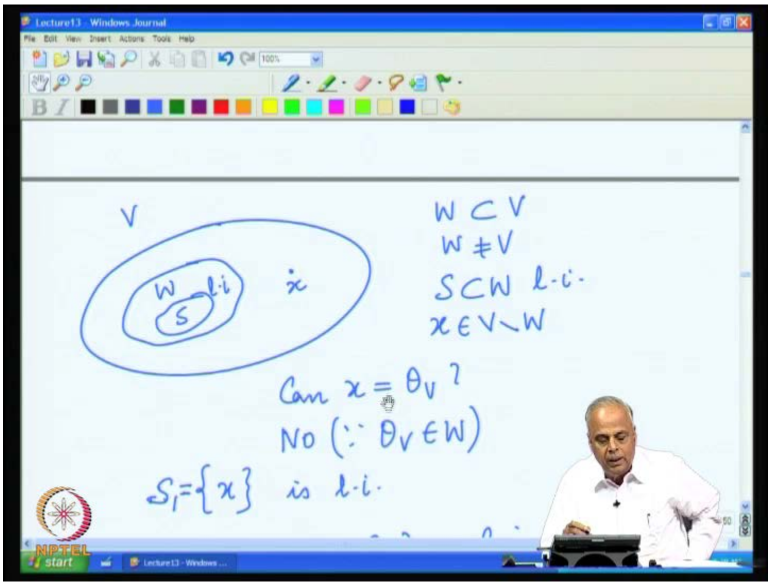

Suppose, we take V and take a true subspace W so W is contained in V and W is not equal to V and we take a S inside the linearly independent set, so S contained W linearly independent. So, we have vector space V inside that sitting a proper subspace W inside that sitting a linearly independent set and W is not equal to V there something outside. So, let us pick up vector x which sitting outside so x belongs V minus W, there is a vector x sitting outside W. Because we assuming the W is not equal to V in that case can x be zero, the answer is no, because every subspace must contain zero vector and so it is zero vector sitting inside this W. No, because theta v belongs W, because W is a subspace it must contain the zero vector so x cannot be zero vector and therefore, the set S 1 consisting only that vector is linearly independent. Now, we have a linearly independent set outside W and we have linearly independent set inside W therefore, by previous analysis these two must be linearly independent.

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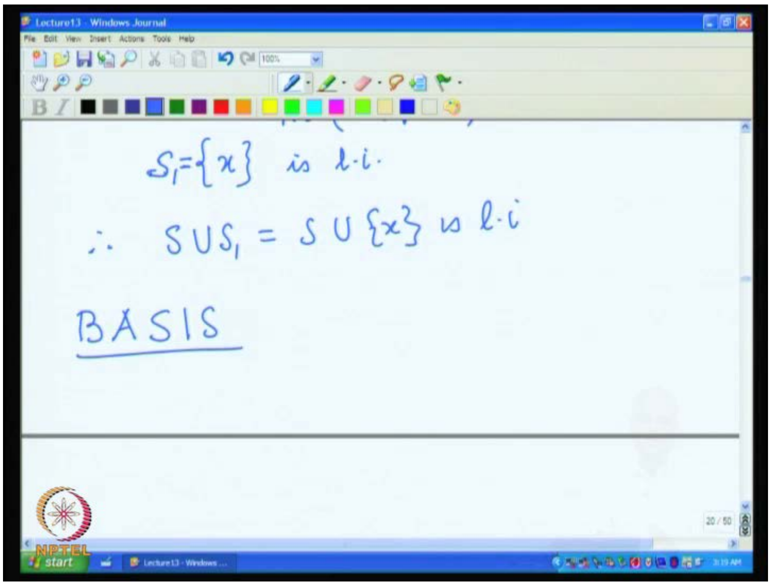

Therefore, S union S 1 that is S union x is linearly independent. So, other word, if you take a subspace is linearly independent vectors and pick some other vector from outside. And appendix to your linearly independent set in W, it still retained it is linearly independent set. So, if you take a vector x outside W appendix to the vector it is a original linearly independent set and it remains to be linearly independent this a very useful (( )). We will use it in basis analysis the next important notion will be introduced in is that of a basis the name such as, is a fundamental notion it is basis of many of your analysis, which will look out in the next class.