Advanced Linear Algebra
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Lecture – 9
Row Space of a Matrix

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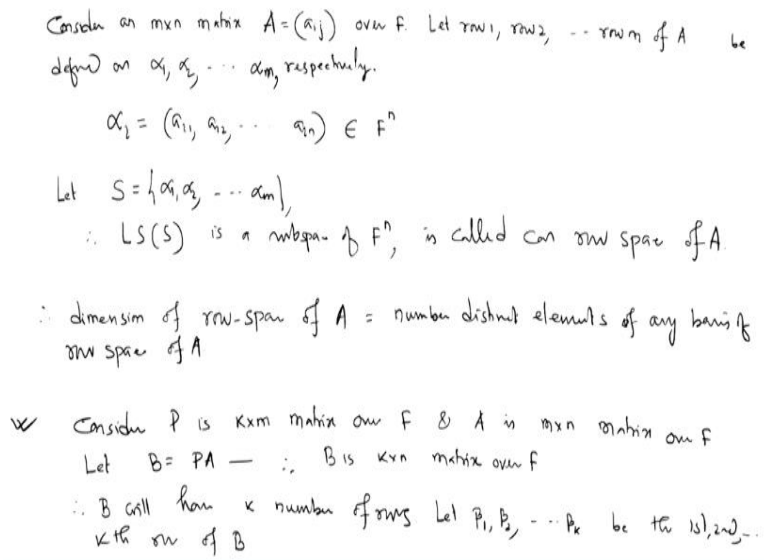

Welcome to lecture series. Today we will discuss about row space of a matrix, so what is that and then we will also discuss about details about it also. Consider an mxn matrix A = (a_{ij}) over F. Let row 1, row 2, row m of A be defined as say {α_1, α_2, ... α_m} respectively. So, I am calling first row of A as α_1, second row of A is α_2, and m-th throw of A is α_m respectively.

So, certainly here alpha i is ith row of A is nothing α_i = {a_{i1}, a_{i2}, ... a_{in}} So, α_i is of this form since we have defined the matrix A over F, so, each entry α_i = {a_{i1}, a_{i2}, ... a_{in}} ∈ F. So, this means that this belongs to also I can say F^n which is n-tuple space over the F. So, I have defined my {α_1, α_2, ... α_m} as m rows of A. Let S = {α_1, α_2, ... α_m}, so linear span of S which is subspace of F^n.

Of course, this linear span of S is also going to be a vector space, it satisfies all the axioms of the vector space, is called as row space of matrix A. So, linear span up all the rows of A basically give you a row space of A which is basically a subspace of the vector space F^n. So, this implies if I want to know what is the, dimension of row space of A = number of distinct elements of any basis of row space of A.

Now to understand how to construct the basis of the row space of A, we need to understand something product of two matrices and the row space for the; if I consider two matrices B and say P such that B = PA, then we want to see the relation between the row space of B and row space of A. This may not be clear, so let me just introduce some elementary matrix to the concept here and then I will come back to basically how to find the basis of a mxn matrix.

So, consider P is kxm matrix over say F and A is mxn matrix over say F. Let B = PA. So B is kxn matrix because P is kxm and A is mxn, so PA will be kxn matrix over F, I mean entries of B is from F. B will have k number of rows. Let $\{\beta_1, \beta_2, \dots \beta_k\}$ be the first, second and k-th row of B.

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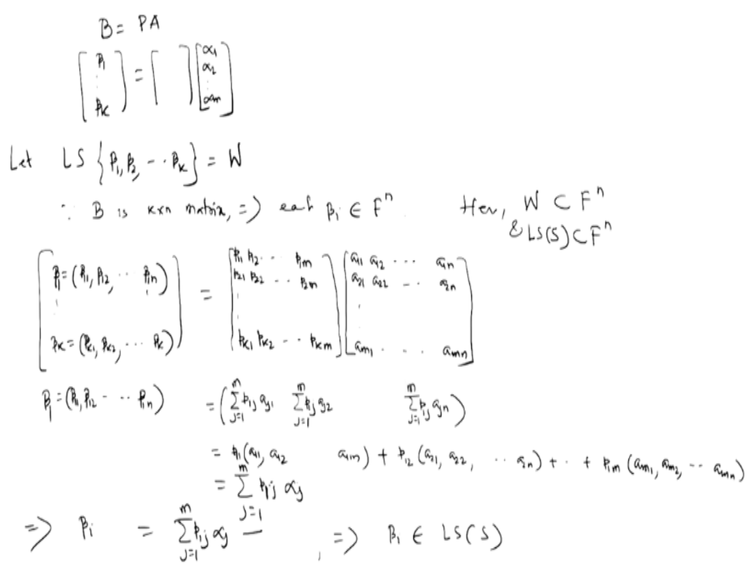

So, what I have considered let me rewrite this one B = PA, where B = $\{\beta_1, \beta_2, \dots \beta_k\}$ this is this row. This P matrix is equal to I have written say P matrix into I have already notated that first row, second row and m-th row of As $\{\alpha_1, \alpha_2, \dots \alpha_m\}$. Let LS(linear span) $\{\beta_1, \beta_2, \dots \beta_k\}$ = W. Since B is kxn matrix implies each $\beta_i \in F^n$ that n-tuple space, as $\alpha_i \in F^n$, here $\beta_i \in F^n$.

Now, we want to see is there any correlation between the subspace W which is basically linear span of $\{\beta_1, \beta_2, \dots \beta_k\}$ and the subspace LS(S), I mean to say subspace generated by rows of the

A matrix. We know both of them are subspaces of the vector space F^n. So, here W is subspace of F^n and LS(S) is also subspace of F^n, this is given to us, I mean from the definition I am getting this one.

Now we want to check whether any correlation between these two subspaces. So, to understand that correlation, I have to also understand the algebraic relationship between these two matrices, I mean relation between B, P, A. So, let me quickly check what is the

$$\begin{bmatrix} \beta_1 = (\beta_{11}, \beta_{12}, \dots \beta_{1n}) \\ \beta_2 = (\beta_{21}, \beta_{22}, \dots \beta_{2n}) \\ \beta_k = (\beta_{k1}, \beta_{k2}, \dots \beta_{kn}) \end{bmatrix} = \begin{bmatrix} (p_{11}, p_{12}, \dots p_{1m}) \\ (p_{21}, p_{22}, \dots p_{2m}) \\ (p_{k1}, p_{k2}, \dots p_{km}) \end{bmatrix} \begin{bmatrix} (a_{11}, a_{12}, \dots a_{1n}) \\ (a_{21}, a_{22}, \dots a_{2n}) \\ (a_{m1}, a_{m2}, \dots a_{mn}) \end{bmatrix}$$

So, this implies if you simply write down the beta 1 when all the n components of beta 1. So, this is equal if write down only simply $\beta_1 = (\beta_{11}, \beta_{12}, \dots \beta_{1n})$ = ($\sum_{j=1}^{m} p_{1j} a_{ji}$ $\sum_{j=1}^{m} p_{1j} a_{j2}$ $\sum_{j=1}^{m} p_{1j} a_{jn}$) = $p_{12}(a_{11}, a_{12}, \dots a_{1n})$ + $p_{12}(a_{21}, a_{22}, \dots a_{2n}) + \dots + p_{1m}(a_{m1}, a_{m2}, \dots a_{mn})$ = $\sum_{j=1}^{m} p_{1j} \alpha_j$.
So, in general I can say that $\beta_i = \sum_{j=1}^{m} p_{ij} \alpha_j$. So, what does it mean?

So, this means that each element $\{\beta_1, \beta_2, \dots \beta_k\}$, I can express as a linear combination of $\{\alpha_1, \alpha_2, \dots \alpha_m\}$. So, this implies what? $\beta_i \in$ LS(S), a subspace generated by S.

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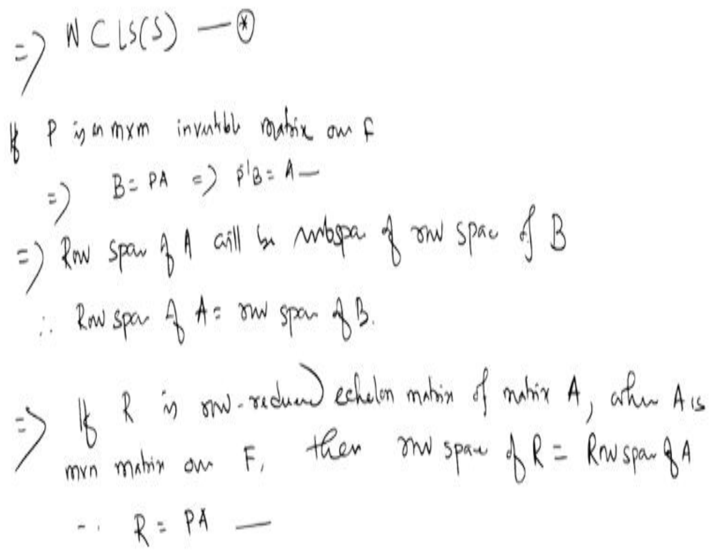

This implies the row space of B, W will be subspace of LS(S). So, subspace generated by the rows of B matrix which is basically equal to P to A it is a subspace of the row space of A. So, this is a nice result, I am getting this one. If P is nxm invertible matrix over F. So, this implies what? My B = PA, this implies that $P^{-1}B = A$.

Now, friends arguing same way I can again show that the row space of A will be subspace of row space of B. So, this implies that row space of A will be subspace of a row space of B. So, we got an interesting result that if we have B = PA where P is an invertible matrix then row space of B and row space of A are same. This implies if R is row-reduced echelon form or even row-reduced matrix also.

If R is a row-reduced echelon matrix of matrix A where A is mxn matrix over F, then row space of R = row space of A since R=PA because R has been obtained multiplying from the left, a sequence of invertible elementary matrices if you denote that is equal to P, then by previous argument I can say that the row space of R is exactly called to row space of A.

So, now we are in the position to find a basis for the row space of given matrix. Instead of talking any mxn matrix over F, I will consider now any mxn row-reduced echelon matrix over F because I know that both if you consider A is the mxn matrix and the corresponding row-reduced equivalent matrix is R, then row space of A equal to row space of R.

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Let A be an $m \times n$ matrix over F. Let R be the corresponding row-reduced echelon matrix. Let R has r number of nonzero rows. Then these nonzero rows will form a basis of the row space of A.

Pf: Since Row space of A is equal to row space of R, so, if B is a basis for R then it is also basis for A.

Let $\rho_1, \rho_2, \dots \rho_r$ are the 1st, 2nd, .. rth nonzero rows of R

Let $B = \{\rho_1, \rho_2, \dots \rho_r\}$ claim B is a basis of row space of R.

Let, the leading entry in row 1, row 2, .. row r appear in column $k_1, k_2, \dots k_r$, respectively.

We know, (a) $R(i,j) = 0$ if $i > r$

(b) $R(i, k_j) = \delta_{ij}$ $1 \le j \le r$, $1 \le i \le r$

(c) $k_1 < k_2 < \dots < k_r$

Consider $\beta = (b_1, b_2, \dots b_n)$ be any element of the row space of R

Ex

$$R = \begin{bmatrix} 1 & 0 & 3 & 0 & 2 \\ 0 & 1 & 2 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$k_1 = 1$, $k_2 = 2$, $k_3 = 4$

Let A be an mxn matrix over F. Let R be the corresponding row-reduced echelon matrix. Let R has r number of nonzero rows. Then these nonzero rows will form a basis of the row space of A. Since row space of A is equal to row space of R, so B is the basis for R then it is also basis for A. So, I will not talk about A, I will stick to only R. Let $\{\rho_1, \rho_2, \dots \rho_r\}$ are the first, second and r-th nonzero rows of R.

Let B = $\{\rho_1, \rho_2, \dots \rho_r\}$. Claim B is a basis of a row space of R. I mean to say any element of row space of R can be written as linear combination of $\{\rho_1, \rho_2, \dots \rho_r\}$. Let the leading entry in $\{\rho_1, \rho_2, \dots \rho_r\}$ appear in column $\{ k_1, k_2, \dots k_r\}$ respectively. Quickly let us recall the definition of row-reduced echelon form. We know, (a) R(i,j) = 0 if i $>$ r.

This is we know the condition for the row-reduced echelon matrix is like that that R(i,j) = 0 if i $>$ r. So, I can parallelly take an example so that we can visualize this concept. So, I will also consider say my R = [1 0 3 0 2 : 0 1 2 0 –1 : 0 0 0 1 0 : 0 0 0 0 0]. So, if I consider this matrix is it a row-reduced echelon matrix that has satisfied all the axioms? Yes.

So, leading entry here k_1 = 1, k_2 =2 and second column, k_1 means column number in which leading entries appear. So leading entry in first row and first column say k_1=1. Leading entry in second nonzero rows appearing on the second column so k_2 = 2, leading entry in third nonzero row appear in fourth column, so k_3 = 4 and the last row is 0 rows, so no question of leading entry displays here. So, here there are three non zeros.

So, for i $>$ 3 will be 0 that is clear from here. Second one your, (b) R(i, k_j) $= \delta_{ij}$, here I mean $1 \leq j \leq r, 1 \leq i \leq r$ because k_1 , k_2 , k_r , $1 \leq i \leq r$, so this also correct and third one your $k_1 < k_2 < k_3 \dots < k_r$. So, this is my condition for a mxn matrix R to be row-reduced echelon form and I can see this is satisfied here also.

Consider $\beta = \{\beta_1, \beta_2, \dots \beta_n\}$ be any element of the row space of R. Then my claim is that beta can be written as a linear combination of $\{\rho_1, \rho_2, \dots \rho_r\}$ and certainly because they are the nonzero rows, so any element of the row space of R is basically a linear combination of $\{\rho_1, \rho_2, \dots \rho_r\}$. only.

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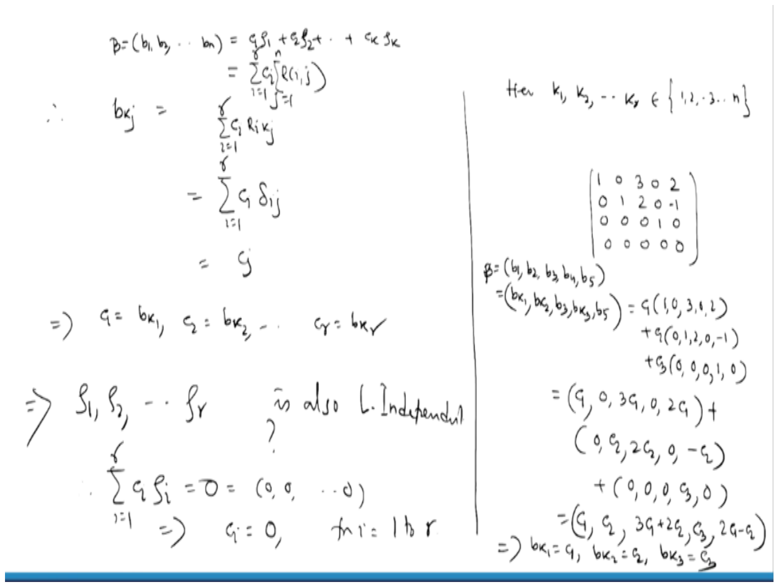

Now, my claim is that if I write down any linear combinations say $\beta = (b_1, b_2, \dots b_m) = c_1\rho_1 + c_2\rho_2 + \dots + c_k\rho_k$ then if we compare the B_{k_i} entry on k i-th entry of the; see here $k_1, k_2, \dots k_r \in \{1, 2, 3, \dots n\}$. They belong to this set k_1 may be 2 or 3 depending on the matrix, so like this only as we have seen previous examples, $k_1 = 1$, $k_2 = 2$, $k_3 = 4$.

So, it is again in the domain of 1, 2, 3, 4 and 5, so they are n = 5. Now, if I say B_{k_i}, B_{k_i} means k i-th entry of my β that is equal to what? So if I take a linear combination of this then they are k-th entry. So, this is basically nothing I can write down $\beta = (b_1, b_2, \dots b_m) = c_1\rho_1 + c_2\rho_2 + \dots + c_k\rho_k = \sum_{i=1}^{r} c_i \sum_{j=1}^{n} R(i,j)$; let me take the example parallelly R = [1 0 3 0 2 : 0 1 2 0 –1 : 0 0 0 1 0 : 0 0 0 0 0].

So, this is my $\beta = (b_1, b_2, b_3, b_4, b_5) = (b_{k_1}, b_{k_2}, b_3, b_{k_3}, b_5) = c_1$ (1 1 0 3 0 2) + c_2 (0 1 2 0 –1) + c_3 (0 0 0 1 0). So, it will be $B_{k_j} = \sum_{i=1}^{r} c_i R_{ik_j} = \sum_{i=1}^{r} c_i \delta_{ij} = c_j$.
and if you take any element beta which is belonging that space, then I am getting that the $c_1 = B_{k_1}$, $c_2 = B_{k_2}$ and $c_r = B_{k_r}$. So, this is implies also $\{\rho_1, \rho_2, \dots \rho_r\}$ is also linearly independent, how?

Because if I consider linear combination this one since sigma $\sum_{i=1}^{r} c_i \rho_i = 0$ = (0 0 0 0) element. This implies that already we have checked that $c_i = 0$. Any element $\beta = (b_1, b_2, \dots b_n)$ which is a linear combination of $\{\rho_1, \rho_2, \dots \rho_k\}$, we have seen that the constant $c_1 = b_{k_1}$, $c_2 = b_{k_2}$ like that.

So, this implies that any linear combination of this which is valid for all the elements, so it also valid for zero element also. So, this implies that c_i = 0 for i = 1 to r. So, this is the result. Now, exactly same thing I can also write here. See the right-hand side is equal to (c_1, 0, 3c_1, 0, 2c_1) + (0, c_2 , 2c_2, 0, -c_2) + (0, 0, 0, c_3, 0) = (c_1, c_2, 3c_1+2c_2, c_3, 2c_1–c_2).

Comparing the coefficient this side c_1 = B_{k_1} , c_2 = B_{k_2} and c_r = B_{k_r}. So, {ρ_1, ρ_2, … ρ_k}, form a basis for the row space of the matrix R or row space of the matrix A. Thank you.