Advanced Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

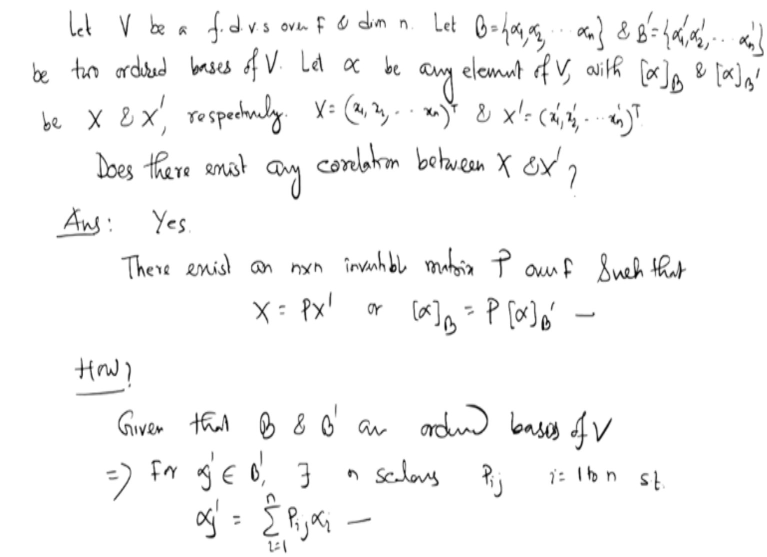
Lecture – 8
Change of Ordered Basis in Finite Dimensional Vector Space

Welcome to lecture series on advanced linear algebra. In my last lecture, I have introduced the concept of basis, dimension and also ordered basis that is basis within an ordered finite sequence of vectors, which also span the vector space itself that is basically ordered basis. By introducing ordered basis in a finite dimensional vector space, we have seen if we consider V is a finite dimensional vector space of dimension n, then as in the case of F^n space of n-tuples where each element basically is of the form of coordinated type of things.

A belongs to the F^n, α is having components like $x_1, x_2, \dots\ x_n$ type there is a coordinate structure you see, its first entry is x_1, second x_2 like that. So, similar coordinate concept can be also introduced by considering ordered basis for any finite dimensional vector space and on the basis of that we defined for a given ordered basis in a given pattern of vector space for an element α a coordinate matrix with respect to that ordered basis.

And we have seen that there is a one-to-one relation between the set of elements of the finite dimensional vector space V, say dimension n over the field sat F to our corresponding vector space F^n. And in terms of coordinate matrix also can see similar relations that for each element α belongs to V, one can have a column matrix over F^{nx1}. Similar to F^n, F^{nx1}, I mean collection of all the column matrix, nx1 column matrix they also follow vector space over F. You can check it easily. So, there is one-to-one relation also.

(Refer Slide Time: 02:53)



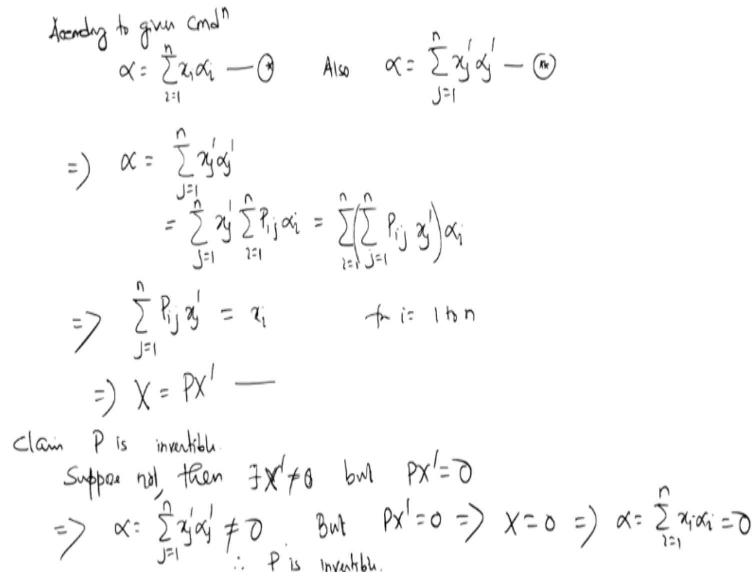
There is one question if I recall correctly let V be a finite dimensional vector space over field F and dimension say n. Let B = $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and B$'$ = $\{\alpha_1', \alpha_2', \ldots \alpha_n'\}$ be two ordered basis of V. Let α be any element of V, with $[\alpha]_B$ & $[\alpha]_{B'}$ be X and X$'$ respectively.

Here X = $(x_1, x_2, \ldots x_n)^T$ and $X' = (x_1', x_2', \ldots x_n')^T$. Now, we are curious to know is there any correlation between X and X$'$? So, question is does there exist any correlation between X and X$'$? Answer is yes. My claim is that there exist an nxn invertible matrix P over the field F.

Such that X = PX' or $[\alpha]_B$ = P $[\alpha]_{B'}$. So, now the question is how? Given that B and B$'$ are ordered basis of V. So, this implies for each element of B prime I will be able to express in terms of linear combination of the elements of P. So, I can say for each $\alpha_j' \in$ B$'$ ∃ n scalars say P_{ij}, i = 1 to n such that your $\alpha_j' = \sum_{i=1}^{n} P_{ij}\, \alpha_i$, , i = 1 to n.

So, I have written α_j' equal to linear combination of the vectors of α_i. Already I have mentioned that the scalar product star though there is no binary operation, I have not mentioned but there is a binary operation between P_{ij} *and* α_i that is the scalar multiplication over the vector space V.

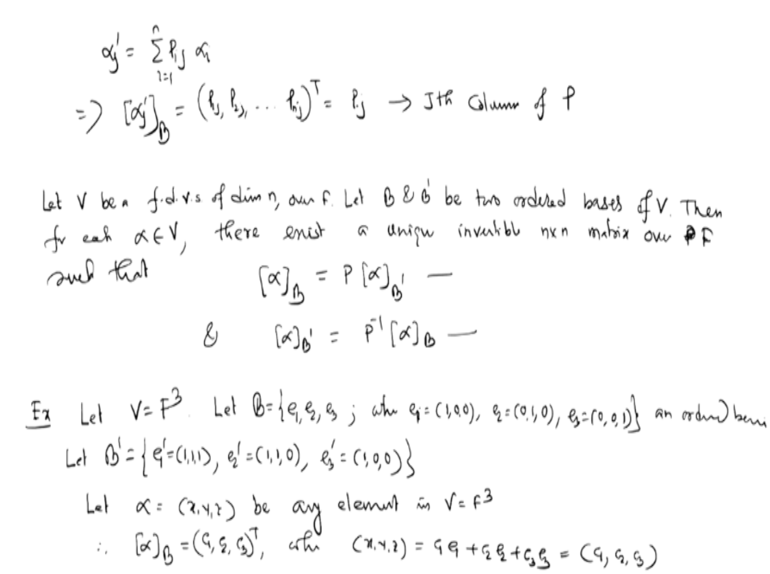
(Refer Slide Time: 07:42)



According to given condition your $\alpha = \sum_{i=1}^{n} x_i \, \alpha_i$, and also $\alpha = \sum_{j=1}^{n} x_j{}' \, \alpha_j{}'$ because it is given to me that $[\alpha]_B$ & $[\alpha]_{B'}$ be X and X′ respectively, so I can write down like this * and ** of this. So, this implies, I can write down $\alpha = \sum_{j=1}^{n} x_j{}' \, \alpha_j{}' = \sum_{j=1}^{n} x_j{}' \, \sum_{j=1}^{n} P_{ij} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij} \, x_j{}' \right) \alpha_i$. This implies $\sum_{j=1}^{n} P_{ij} \, x_j{}' = x_i$ for i=1 to n, this implies X= PX′. Now, claim is be P is invertible.

Suppose not, then we know if P is not an invertible matrix, then $\exists$ X′ ≠ 0, but P X′= 0 So, this implies $\alpha = \sum_{j=1}^{n} x_j{}' \, \alpha_j{}'$, here this will not be equal to 0, but P X′ = 0, implies X = 0. So, X = 0 implies, $\alpha = \sum_{i=1}^{n} x_i \, \alpha_i = 0$. So, in one end I am getting nonzero, another end I am getting 0. So, this is not possible, so that P is not invertible is not possible, so P is invertible.

(Refer Slide Time: 12:47)



We have seen each $\alpha_j{}' = \sum_{i=1}^{n} P_{ij}\ \alpha_i$. This implies the $[\alpha_j{}'\ ]_B = (P_{1j}, P_{2j}, \dots P_{nj})^T$ So, this is basically I can write down the jth column of P matrix, so this is with respect to B. So, let me conclude it. Let V be a finite dimensional vector space of dimension n over F. Let B and B′ be two ordered basis of V.

Then for each α belongs to B, there exist a unique invertible nxn matrix over F such that $[\alpha]_B$ = P $[\alpha]_{B'}$ and $[\alpha]_{B'}$ = P^{-1} $[\alpha]_B$ So, this is basically conclusion of this result. So, let me take some examples. Before taking examples, now the next question is like this.

Now suppose V is a finite dimensional vector space over the field say F and B be an ordered basis for V, this is given to us. Now, suppose a nxn invertible matrix P is also given to us then does there exist an ordered basis B′ such that $[\alpha]_B$ = P $[\alpha]_{B'}$ say that the question.

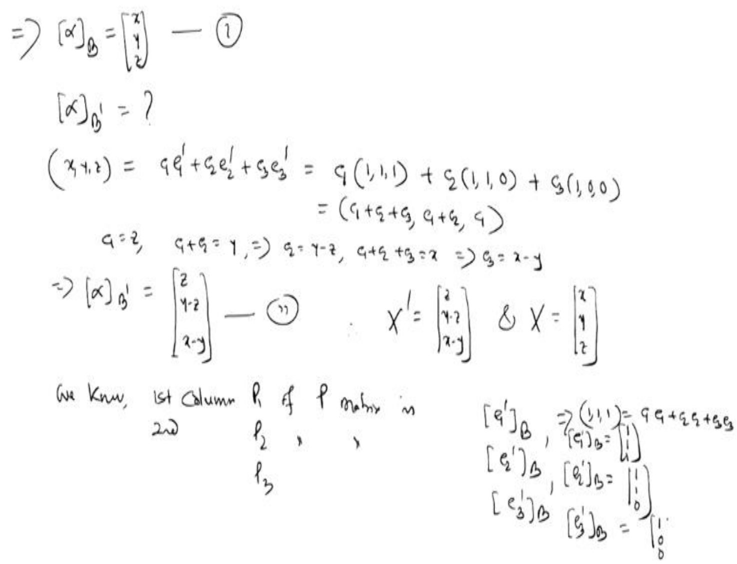
Answer is again yes, one can have a second ordered basis if an invertible nxn matrix is given to us, which of course has to be defined over the corresponding field. So, let me take an example first. Let V =F^3, I mean space of 3-tuples over the field F. For sake of simplicity, I can take F equal to real number. Let B={e_1, e_2, e_3 where e_1 = (1, 0, 0), e_2 = (0, 1, 0) and e_3 = (0, 0, 1)}.

So, I have taken standard basis for the F^3. Since I am saying that the first element is e_1, second e_2, third e_3. I am also saying this in ordered basis. So this B an ordered basis. Let B' ={ $e_1{}'$ = (1, 1, 1) $e_2{}'$ = (1, 1, 0), $e_3{}'$ = (1, 0, 0). One can immediately check this B' is a basis for the space of F^3.

Also because if you see as the linear combination of $c_1 e_1{}'$ + $c_2\ e_2{}'$ + $c_3\ e_3{}'$ = 0, if you consider for any c_1, c_2, c_3 such that $c_1 = 0, c_2 = 0, c_3$ = 0 the only solutions and that can be also trivially obtained because the coefficient matrix will be invertible matrix, you can check it. So, this is given to us B primary also an ordered basis. Let α = (x, y, z) be any element in V = F^3, so this one.

So, what about the, $[\alpha]_B = (c_1, c_2, c_3)'$ where (x, y, z)= $c_1\ e_1$ + $c_2\ e_2$ + $c_3\ e_3$ = (c_1, c_2, c_3), you can see because e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1). So if I simplify then I am getting $c_1\ e_1$ + $c_2\ e_2$ + $c_3\ e_3$ = (c_1, c_2, c_3).

(Refer Slide Time: 19:45)



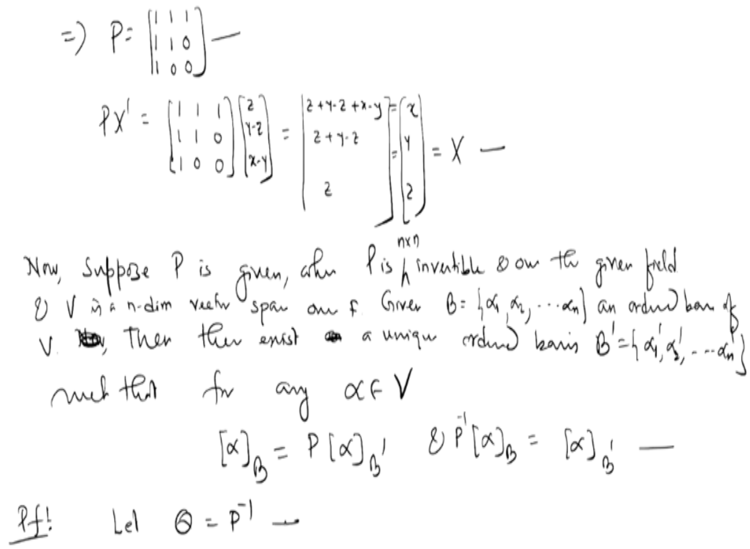
This implies the $[\alpha]_B$ = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, this one. What can you say about the $[\alpha]_{B'}$? So, I do not know, let me quickly calculate it. So, I know (x, y, z) = $c_1 e_1{}'$ + $c_2\, e_2{}'$ + $c_3\, e_3{}'$ = c_1 (1, 1, 1) + c_2 (1, 1, 0) + c_3 (1, 0, 0) = $(c_1 + c_2 + c_3,\ c_1 + c_2,\ c_1)$, because I have to find out what is $c_1,\ c_2,\ c_3$. Once this is known then the transpose of this coordinate matrix will basically give me the $[\alpha]_{B'}$. So, comparing this I am getting the equation for c_1 = z and $c_1 + c_2$ = y. This implies that your c_2 = y – z and $c_1 + c_2 + c_3$ = x implies that c_3= x –y.

So, this implies $[\alpha]_{B'}$ will be what c_1 = z, c_2 = y – z , c_3= x –y. So, my definition of the coordinated matrix of a vector I have the coordinate matrix of the vector α because both the ordered basis I have already derived. Now, we want to check by that relation that there exists a nonsingular matrix P such that X = P X′, let us see it. So, my X′ = $\begin{bmatrix} z \\ y - z \\ x - y \end{bmatrix}$ and X = $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ okay it is fine. Now, I have to find out the P matrix. According to definition of the P matrix, we know the first column P_1 of P matrix is basically $[e_1{}']_B$ that will be my first column of P and second column P_2 of P will be coordinate matrix of $[e_2{}']_B$and similarly P_3 of P will be $[e_3{}']_B$.

This is equal to how much? See $e_1{}'$, so if I write down this implies (1 1 1) = $c_1\, e_1$ + $c_2\, e_2$ + $c_3\, e_3$. So, this implies e_1 =(1 0 0), e_2 = (0 1 0), e_3= (0 0 1). So, the $[e_1{}']_B$ = $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, And $[e_2{}']_B$ = $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and

$[e_3']_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(Refer Slide Time: 24:32)

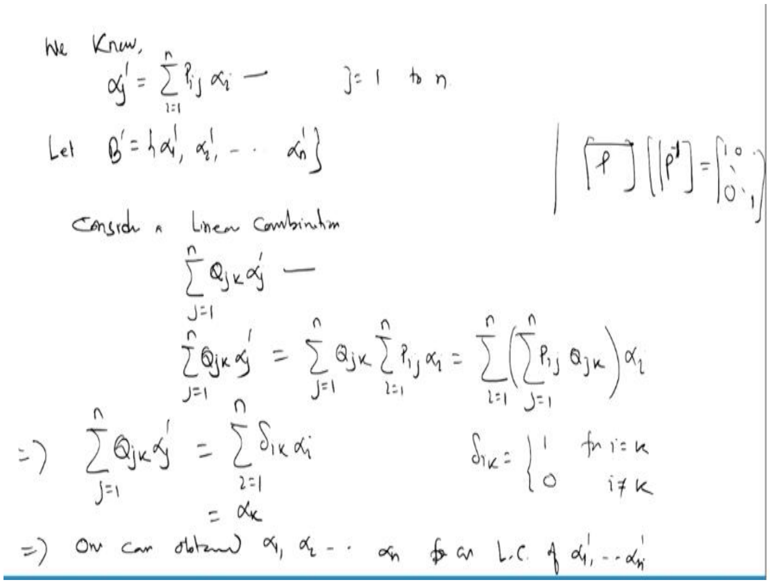


So, this implies my P= $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. So, this is an invertible matrix which I can quickly check it. Now, I want to cross check whether my X = P X′ or not. Now, if I say X = P X′ = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} z \\ y-z \\ x-y \end{bmatrix} = \begin{bmatrix} z+y-z+x-y \\ z+y-z \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ =X. So, there may be some error, I do not know, but you have to cross check, simply do not trust on my this algebra, you have to also cross check and see whatever I have written correct.

Let me see it. So, this is basically z + y – z + x – y = x, z + y – z = y, z this one. So, I am getting basically like this. This implies that I am getting this is equal to, I can write down this is equal to X. So, P X′ = X. Now suppose P is given where P is invertible and over the given field, how to find the second ordered basis when one ordered basis is given the space and it is also given B = {α_1, α_2, ... α_n} an ordered basis of V.

Then there exists a unique ordered basis B′ = { α_1', α_2',… α_n'} such that α ∈ V, $[\alpha]_B$ = P $[\alpha]_{B'}$ & P^{-1} $[\alpha]_B = [\alpha]_{B'}$, now how to prove that one also? This proof is again not difficult one. So, the proof is like this. Let Q = P^{-1} , because P is given to me invertible, so I can write Q = P^{-1}, like this thing.

(Refer Slide Time: 28:30)



We know $\alpha_j{}' = \sum_{i=1}^{n} P_{ij}\, \alpha_i$, so j = 1 to n. So, I can have $\alpha_1{}'$, $\alpha_2{}', \ldots\ \alpha_n{}'$ up to this thing. Let $\mathrm{B}' = \{\alpha_1{}', \alpha_2{}', \ldots\ \alpha_n{}'\}$. So, I have constructed n elements of B. I claim this is an ordered basis for the vector space V with this order first element is my $\alpha_1{}'$, $\alpha_2{}', \ldots\ \alpha_n{}'$.

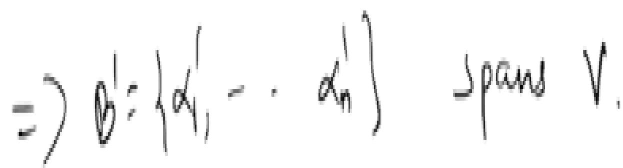
If I somehow can show that these n elements span your vector space V that I am through because the dimension of the vector space V is n, so we need for any basis exactly n number of linearly independent elements. There are exactly n elements. Now, if I show that they span the space V, then I am through. So, what I will do consider a linear combination like $\sum_{j=1}^{n} Q_{jk}\, \alpha_j{}'$.

If somehow I can show that this linear combination give me an element from B then also I am through. Now, this $\sum_{j=1}^{n} Q_{jk}\, \alpha_j{}' = \sum_{j=1}^{n} Q_{jk} \sum_{i=1}^{n} P_{ij}\, \alpha_i = \sum_{j=1}^{n}(\sum_{i=1}^{n} P_{ij} Q_{jk})\, \alpha_i$

Now, this implies to Q is inverse matrix of P. So, if I consider P and P inverse then it will be only first row multiplied by first column will have only first entry equal to 1, all other 0. Similarly, second row multiplied to the first one it will basically give me 0 since P into P inverse equal to you know identity matrix which is 1 0 0 like this 1 0, so definitely it has to follow like this.

So, this implies that I can write down this is equal to simply, $\sum_{j=1}^{n} Q_{jk}\, \alpha_j{}' = \sum_{i=1}^{n} \delta_{ik}\, \alpha_i$, $\delta_{ik} = 1$ for i = k, $\delta_{ik} = 0$ for i $\neq$ k. So, this means that this is going to be give me simply α_k. So, linear combinations of α_j' give me α_k also. I mean linear combination α j-th prime coefficient as Q_{j1} , j = 1 to n. So, this implies one can obtain $(\alpha_1, \alpha_2, \ldots \alpha_n)$ as a linear combination of $(\alpha_1', \alpha_2', \ldots \alpha_n')$.

(Refer Slide Time: 33:34)



So, this implies $B' = \{\alpha_1', \alpha_2', \ldots \alpha_n'\}$ span V. So, what we have learned today? We have learnt that if we consider two different ordered basis for a finite dimensional vector space, then the coordinate matrices of any vector α are related by some nonsingular invertible matrix P such that coordinate matrix of α with respect to B equal to p times coordinate matrix of α with respect to second basis.

And an invertible matrix is given which defined on same field basically always vector space is defined, then also when a one ordered basis is given, one can have another ordered basis. Thank you.