Advanced Linear Algebra
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Lecture – 7
Basis and Dimension Part - 2

Welcome to lecture series. In my previous classes, I have discussed about the concept of basis, linear independence and dimension. So, let me continue with that.

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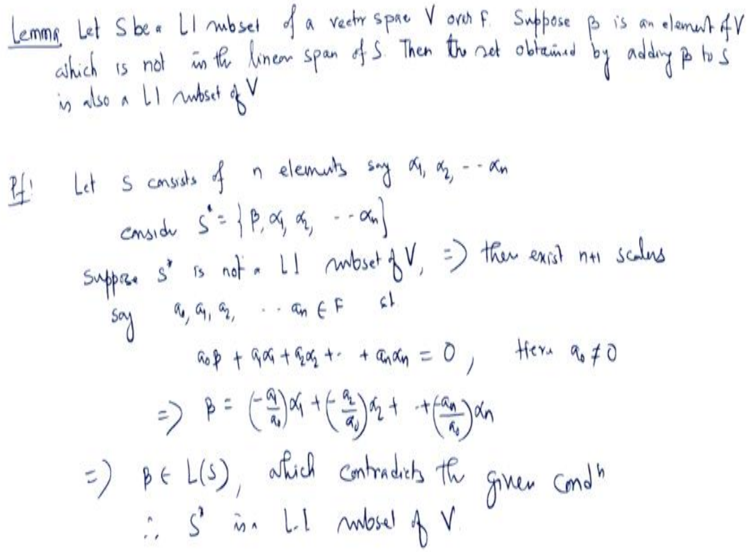

I have shown that if V is the vector space and if it is a finite dimensional and if it spanned by a finite number of elements say $\{\beta_1, \beta_2, \dots \beta_m\}$ then the number of elements of that linearly independent subset will be less than or equal to m. I mean it cannot exceed m. So, there is another small result based on that principle only I will say it. Suppose S be a linearly independent subset of a vector space over F.

Suppose beta is an element of V, which is not in the linear span of capital S. Then by adding beta to capital S will be again a linearly independent subset of V. So, let me get the proof. Let S consists of n elements say $\{\alpha_1, \alpha_2, \dots \alpha_n\}$; they are all distinct elements. Consider $S^*= \{\beta, \alpha_1, \alpha_2, \dots \alpha_n\}$. It is given to me beta is not in the linear span of S. Now, my claim is that; is that is a linearly independent subset of V.

Suppose not, suppose S^* is not a linearly independent subset of V. So, this means that there exist n+1 scalars say not $(a_0, a_1, a_2, \dots\ a_n)\ \in$ F such that $a_0\, \beta + a_1\alpha_1 + a_2\alpha_2 +\ \dots + \alpha_n a_n$ = 0, here $a_0 \neq$ 0, otherwise, β has not come into the picture and we know that a_1 to a_n is linearly independent. So, therefore I have considered $a_0 \neq 0$.

So, this implies that $\beta = \left(\frac{-a_1}{a_0}\right)\alpha_1 + \left(\frac{-a_2}{a_0}\right)\alpha_2 + \cdots + \left(\frac{-a_n}{a_0}\right)\alpha_n$. So, this implies $\beta \in$ L(S), which contradicts the given condition. So S^* is a linearly independent subset of V. Now, this concept will help us from a linearly independent subset of a vector space one can extend this linearly independent subset of the vector space to be basis of the corresponding space.

So, it is something called the extension theorems. So, this we can easily prove by adding one by one elements and so this is I am giving as homework, so that I am saying of course is an extension theorem. What is extension theorem? Suppose V be a finite dimensional vector space over the field say F and S be a linearly independent subset of V, then that subset can be extended to be basis of the corresponding space V also. So, this I am leaving as a homework.

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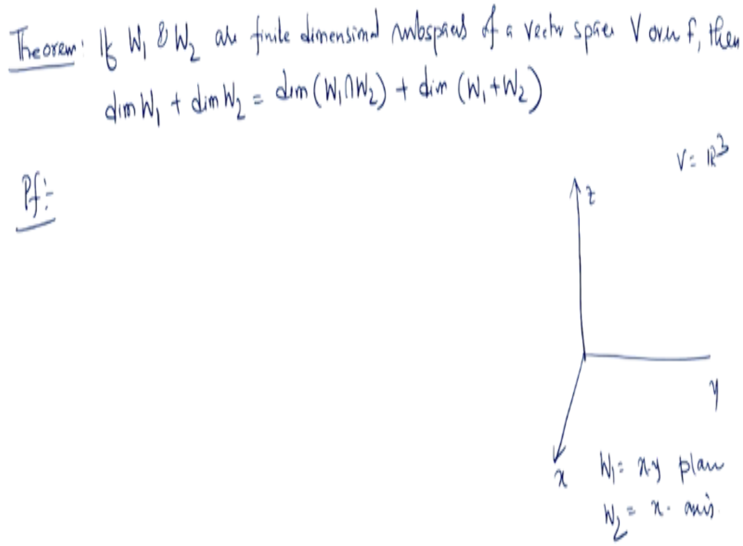

Now, let me consider another interesting subset or subspace that basically come as a part of the sum of subspaces what we introduced in last classes. Now, it is like this. If W_1 & W_2 are finite dimensional spaces of a vector space V over F, then dim W_1+ dim W_2 = dim ($W_1 \cap W_2$)+ dim (W_1 + W_2). Before proving this result, let me quickly give in a geometrical picture so that one can have idea how to proceed to prove this one.

So, what I will do let me take V = R 3. Now, let me consider here say W 1 is equal to this is my x, y, z. So, W_1 is x-y plane and W_2 is x-axis. We know both of them are subspaces. Now, if I consider W_1 + W_2, the sum of actually W_1, W_2 this will give me what? It will give me simply x-y plane. So, dim(W_1 + W_2) = 2 and dim (W_1) =2, dim (W_2) = 1. What about ($W_1 \cap W_2$)?.

($W_1 \cap W_2$) is basically x-axis and dimension of x-axis is again 1. So, you see that dimension of W_1 in this example is equal to 2, dimension of W_2 example 1, so 2 + 1 = 3 and if I see that right hand side dimension of ($W_1 \cap W_2$) that is equal to 1 and plus dimension of W_1 + W_2 is again 2 because W_1 + W_2 is nothing my x-y plane that is W_1 only. So, therefore I see that this relation holds good.

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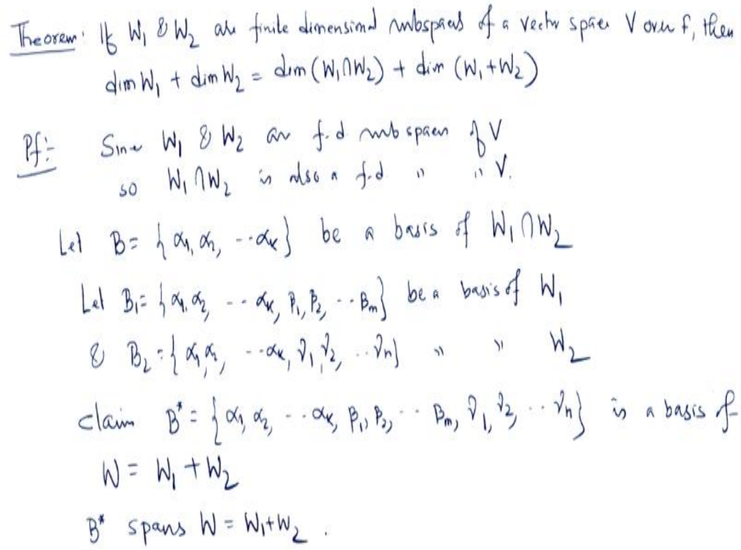

So, now let me give it more generalized form. Since W_1 and W_2 are finite dimensional subspaces of V, so ($W_1 \cap W_2$) is also a finite dimensional subspace of V. Let B = $\{\alpha_1, \alpha_2, \dots \alpha_k\}$ be a basis of ($W_1 \cap W_2$). We see that B is basically a subset of ($W_1 \cap W_2$)and linear span of B = ($W_1 \cap W_2$) and ($W_1 \cap W_2$) is a subset of W_1 and also subset to W_2 .

So, by extension theorems, I can extend the B to B_1 to be basis of the W_1. So, let $B_1 = \{\alpha_1, \alpha_2, \dots \alpha_k, \beta_1, \beta_2, \dots \beta_m\}$ be a basis of the W_1 & B_2= $\{\alpha_1, \alpha_2, \dots \alpha_k, \gamma_1, \gamma_2, \dots \gamma_n\}$ be a basis of W_2. Claim; B^* = = $\{\alpha_1, \alpha_2, \dots \alpha_k, \beta_1, \beta_2, \dots \beta_m, \gamma_1, \gamma_2, \dots \gamma_n\}$ claim this is a basis of sum of subspaces W_1 and W_2, that is W = $W_1 + W_2$.

So, first I have to show that B^* span W and second I have to prove that B^* is linearly independent.

So, first is B^* spans W = $W_1 + W_2$. To prove that one what I have to do?

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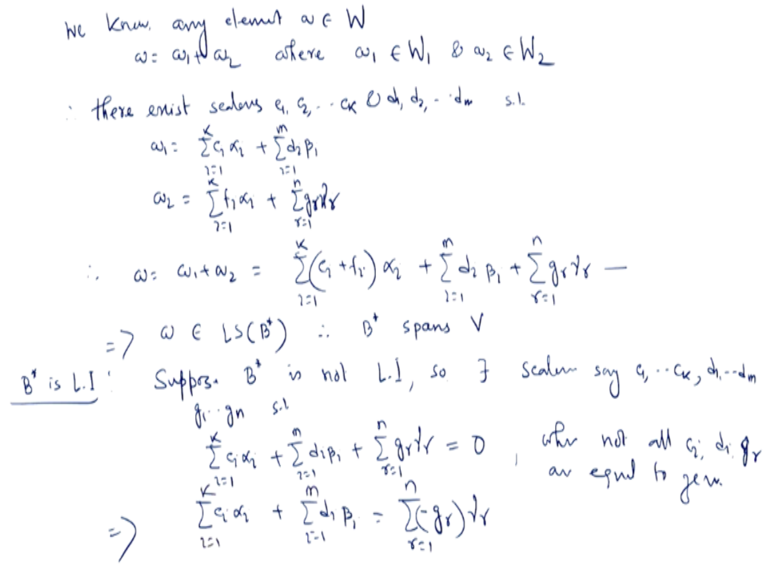

We know any element w ∈ W can be written as w is equal to something like w = $w_1 + w_2$.where $w_1 \in W_1$ and $w_2 \in W_2$. So, since $w_1 \in W_1$, so I can write down the small w_1 as a linear combination of the basis element of w one. So, there exist scalars $c_1, c_2, \dots c_k$ & $d_1, d_2, \dots d_m$ such that $w_1 = \sum_{i=1}^{k} c_i\, \alpha_i + \sum_{i=1}^{m} d_i\, \beta_i$, $w_2 = \sum_{i=1}^{k} f_i\, \alpha_i + \sum_{r=1}^{n} g_r\, \gamma_r$. So, w = $w_1 + w_2 = \sum_{i=1}^{k} (c_i + f_i)\, \alpha_i + \sum_{i=1}^{m} d_i\, \beta_i + \sum_{r=1}^{n} g_r\, \gamma_r$.

So, this means that w can be written as a linear combination of α_i, β_i and γ_i also. So, this implies that w ∈ LS(B^*) or B^* spans V. Now claim is that B^* is also linearly independent. B^* is linearly independent, I have to prove that one. I will prove it by contradiction, suppose it is not, it is not linearly independent. Suppose B stat is not linearly independent, so it is linearly dependent.

So, there exist scalars say $c_1, c_2, \dots c_k$, $d_1, d_2, \dots d_m$, & $g_1, g_2, \dots g_m$ such that $\sum_{i=1}^{k} c_i\, \alpha_i + \sum_{i=1}^{m} d_i\, \beta_i + \sum_{r=1}^{n} g_r\, \gamma_r = 0$. Where not all c_i, d_i, g_r are equal to zero. So, this implies that one can write down sigma $\sum_{i=1}^{k} c_i\, \alpha_i + \sum_{i=1}^{m} d_i\, \beta_i = \sum_{r=1}^{n} (-g_r)\, \gamma_r$.

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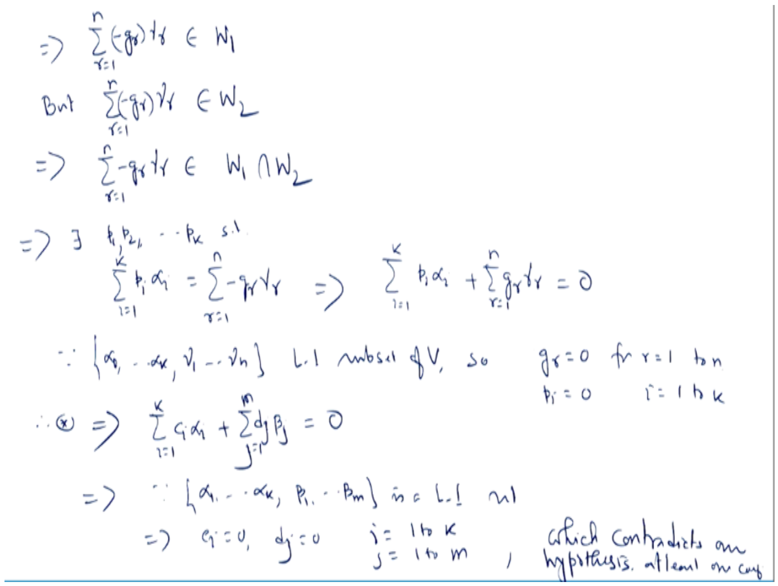

So, this implies $\sum_{r=1}^{n}(-g_r)\,\gamma_r \in W_1$, but $\sum_{r=1}^{n}(-g_r)\,\gamma_r \in W_2$ because $\gamma_1, \gamma_2, \dots \gamma_n$ is basically part of the basis element of W_2. So, this implies what sigma $\sum_{r=1}^{n}(-g_r)\,\gamma_r \in (W_1 \cap W_2)$. So, this belongs this quantity. So, this means that I will be able to express this element as a linear combination of the basis element of $(W_1 \cap W_2)$.

So, this implies $\exists\; p_1,\; p_2,\; \dots\; p_k$ s.t. $\sum_{i=1}^{k} p_i\,\alpha_i = \sum_{r=1}^{n} -g_r\,\gamma_r$. So this implies $\sum_{i=1}^{k} p_i\,\alpha_i + \sum_{r=1}^{n} -g_r\,\gamma_r = 0$. Since $\{\alpha_1, \alpha_2, \dots \alpha_k, \gamma_1, \gamma_2, \dots \gamma_n\}$ is a linearly independent set subset of V, so each $g_r = 0$ for r = 1 to n. Of course, $p_i = 0$, i = 1 to k. So, if I use the $g_r = 0$ for r = 1 to n in my expressions that where I have considered the linearly dependent this relation in star.

So, (*) implies $\sum_{i=1}^{k} c_i\,\alpha_i + \sum_{j=1}^{m} d_j\,\beta_j = 0$. So, this implies again since $= \{\alpha_1, \alpha_2, \dots \alpha_k, \beta_1, \beta_2, \dots \beta_m\}$ is a linearly independent set. So, this implies that c_i = 0 and d_j = 0, I can say this one j = 1 to m so that will be not to confuse, for i = 1 to k and j = 1 to m which contradicts our hypothesis that at least one coefficient not equal to 0.

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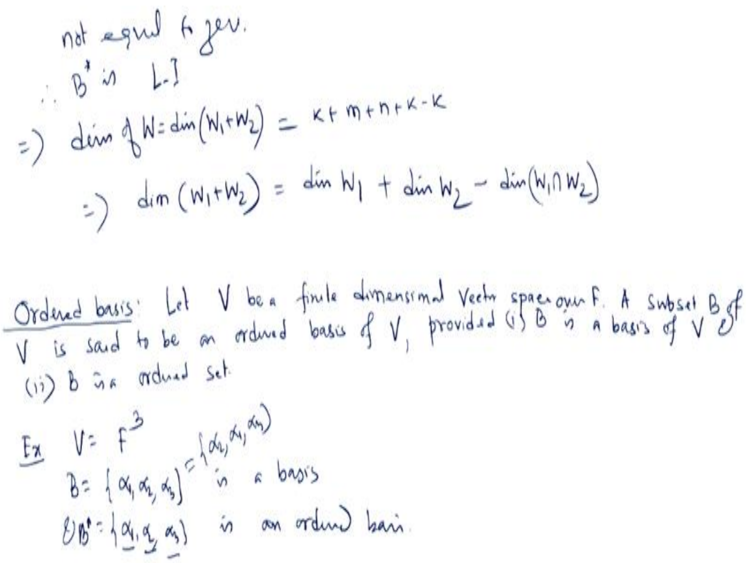

So, B^* is linearly independent. So, this implies that dim(W) = dim(W_1 + W_2) = k + m + n +k -k so this implies dim(W_1 + W_2) = dim W_2 + dim W_2 – dim($W_1 \cap W_2$).

See so far what we have discussed here, we have considered linearly independent or dependent of set but we have not considered the linearly independent or dependent of some finite sequence of vectors. We need the sequence concept to understand about difference of the vector space and its applications. So, I will introduce one terminology called ordered basis. Let V be a finite dimensional vector space over F.

A subset B of V is set to be an ordered basis of V provided one B is a basis of V and second B is an ordered set, I mean to say it follow a pattern, I mean some order is there. So, let me take examples. Now, let me consider V = F^3 . Suppose B = = {α_1, α_2, α_3} is a basis and B^* = {α_1, α_2, α_3} is an ordered basis. So, what is the difference between these things?

See this first B it is a basis, fine, but there is no order whether α_1 will be first element, α_2 will be second element, is it like that? No, see here this B is basically same, I can write this is also equal to α_2, α_1, α_3. But in the case of ordered which is here the first element is fixed, the α_1 is the first element, α_2 is the second element and α_3 is the third element. So, here order is preserved. So let me see what sort of benefit we will see by introducing this concept.

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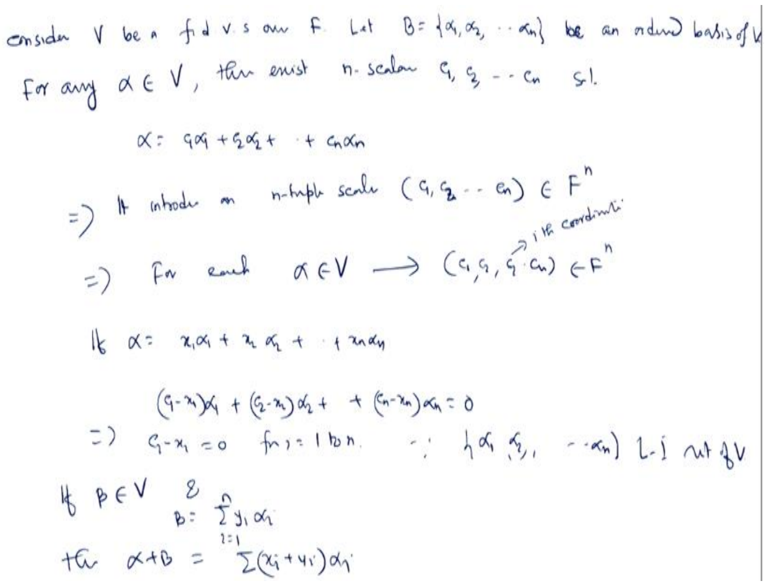

Consider V be a finite dimensional vector space over F. Let B = $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is an ordered basis of V. Now, for any $\alpha \in$ V, there exist n scalars = $\{c_1, c_2, \ldots c_n\}$ such that $\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots + c_n\alpha_n$. So, this implies it introduced an n-tuple scalar that is $c_1, c_2, \ldots c_n \in F^n$. So, this implies for each $\alpha \in$ V there exist an n-tuple say $c_1, c_2, \ldots c_n \in F^n$.

Is it possible that one can have another n-tuple which also express alpha? Let me see. If $\alpha = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n$, this implies that I will have $(c_1 - x_1)\alpha_1 + (c_2 - x_2)\alpha_2 + \ldots + (c_n - x_n)\alpha_n = 0$. This implies $(c_i - x_i) = 0$ for i = 1 to n since $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ this is a linearly independent set. So, this representation is also unique.

And another interesting fact you see if $\beta \in$ V & $\beta = \sum_{i=1}^{n} y_i\, \alpha_i$, then $\alpha + \beta = \sum_{i=1}^{n}(x_i + y_i)\alpha_i$. So, this x_i what I am writing here for the alpha is equal to like this it is introducing the coordinate, ith coordinate. So, here this c_i is ith coordinate.

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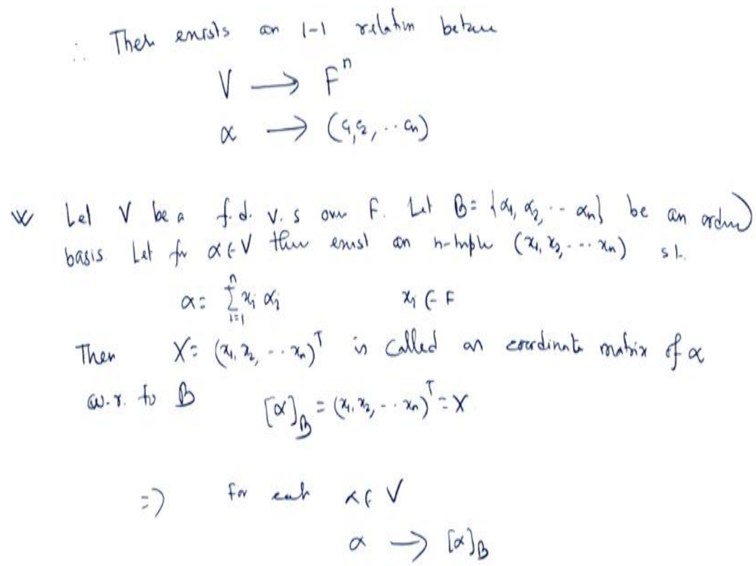

So, there exist a 1-1 relation between the n dimensional vector space V → F^n, α → $(c_1, c_2, \ldots c_n)$. Now, we want to see what is the benefit by introducing this concept? Let V again be a finite dimensional vector space over a field say F. Let B = $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be an ordered basis. Let for $\alpha \in$ V there exist an n-tuple say $(x_1, x_2, \ldots x_n)$ such that your $\alpha = \sum_{i=1}^{n} x_i \, \alpha_i, \quad x_i \in$ F.

Then X = $(x_1, x_2, \ldots x_n)^T$ is called as coordinate matrix of α with respect to ordered basis B. So, the $(x_1, x_2, \ldots x_n)^T$ that coefficients is called the coordinate matrix of α with respect to B. Symbolically I can write that $[\alpha]_B = (x_1, x_2, \ldots x_n)^T$ = X. So, this one belongs to F^{nx1} column matrix.

So, this implies for each $\alpha \in$ V there exist a coordinate matrix say α like this thing with 1-1 map between the set of elements of V and space of n-tuple. So, if you consider any n-tuples say $(x_1, x_2, \ldots x_n) \in F^n$. we have seen that the sigma x i alpha will be element of B. Now this $(x_1, x_2, \ldots x_n)^T$, I am saying a coordinate matrix. So, I can say that for each α there exists a coordinate matrix which 1-1 relation holds good.

Now, we are curious to see if I change the basis, so there are two basis, B_1 and B_2, then how the coordinate matrices are related or if at all there is a relation or not?

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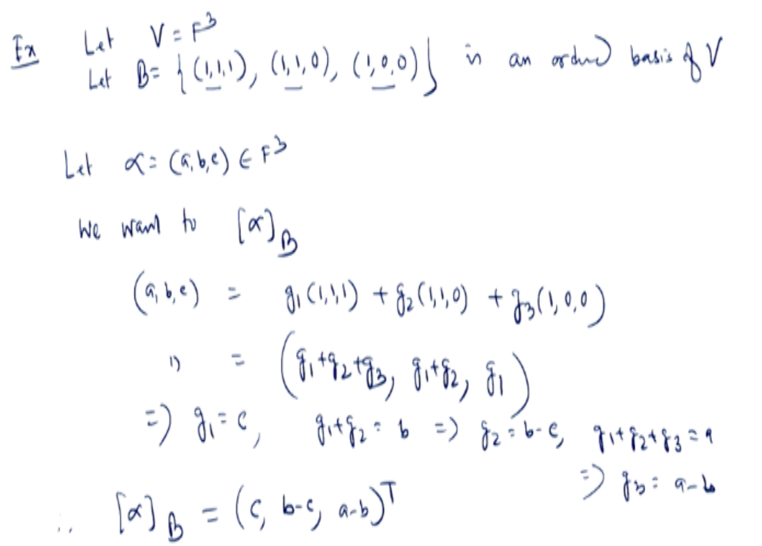

Before that let me take an example of how to find a coordinator matrix of a vector. So, let V $=F^3$. Let B = {(1, 1, 1), (1, 1, 0), (1, 0, 0)}. We can immediately check this B is the basis for the space V. Let me consider this B is also an ordered basis of V. Let α = (a, b, c) ∈ F^3, be any element. Then we want to know what are the coordinate matrix of alpha with respect to this ordered basis B. So, here my first element is this one, second is this one, third this one.

So, I have to write down (a, b, c) = g_1 (1, 1, 1) + g_2 (1, 1, 0) + g_3 (1, 0, 0) = ($g_1 + g_3 + g_3$, $g_1 + g_2$, g_1) which transpose will be basically my coordinate matrix of α. This is equal to $(g_1 + g_3 + g_3,\ g_1 + g_2,\ g_1)$. So, this implies g_1 = c and $g_1 + g_2$ = b implies that g_2 = b – c , $g_1 + g_3 + g_3$ = a. So, this implies that g_3 = a – b. So, $[\alpha]_B = (c,\ b-c,\ a-b)^T$.

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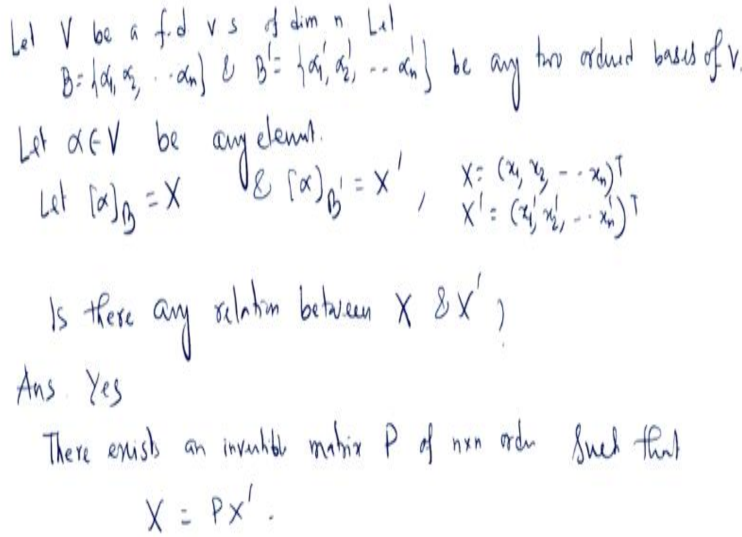

So, now let me prove the question. Let V be a finite dimensional vector space of dimension say n. Let B = $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ and $B' = \{\alpha_1', \alpha_2', \ldots \alpha_n'\}$ be any two ordered basis of V. Let $\alpha \in V$ be any element. Let $[\alpha]_B = X$ & $[\alpha]_{B'} = X'$. I mean X = $(x_1, x_2, \ldots x_n)^T$ and $X' = (x_1', x_2', \ldots x_n')^T$. Now, we want to know is there any relation between X and X' that is the question. Answer is yes. What is that relation? There exist an invertible matrix P of nxn order such that X = PX'. So, now we have to prove for these relations, I will do in the next class. So, I hope you have understood the part which I have covered in this class. So, based on this, I will continue in the next class. Thank you.