Advance Linear Algebra

Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 60
Forms on Inner Product Spaces

Welcome to lecture series Advance Linear Algebra, Today I am going talk about a new terminology Forms on Inner Product Space.

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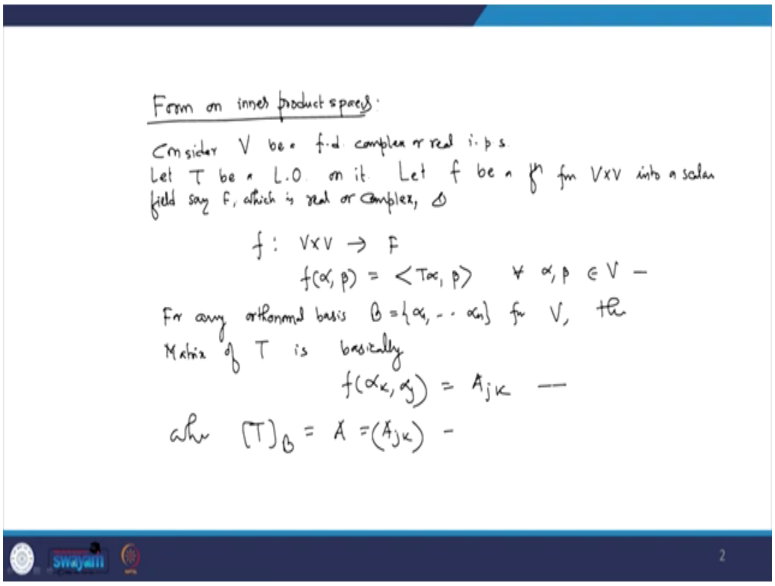

So, let me first talk about that what is the motivation behind introducing this terminology? Consider V be a finite-dimensional complex or real inner product space. Let T be a linear operator on it. Let f be a function, f: VxV → F which is real or complex and f is defined such that f$(\alpha, \beta) = < T\alpha, \beta >$ for every $\alpha, \beta \in$ V

So, if I define a function like this, for this function will see that for any orthonormal basis B = $\{\alpha_1, \alpha_2, \dots\ \alpha_n\}$ for V, the matrix of T is basically can be written as f(α_k, α_j) = A_{jk} where $[T]_B$ = A = (A_{jk}). Then will able to express the matrix representation of the operator T in this function form.

In fact, many questions related to the operator T is basically equivalent to the questions due to the function f and f in fact, determines the upper **(()) (03:43)**. So, it is indirect way is another way to express the opportunity. Sometimes you will see that to talk about the characters to the operator. It will be easy if I talk in terms of this function form. So that is why I am introducing form on an inner product space.

In fact, form can be introduced even other non-linear spaces but here we are interested on inner product space.

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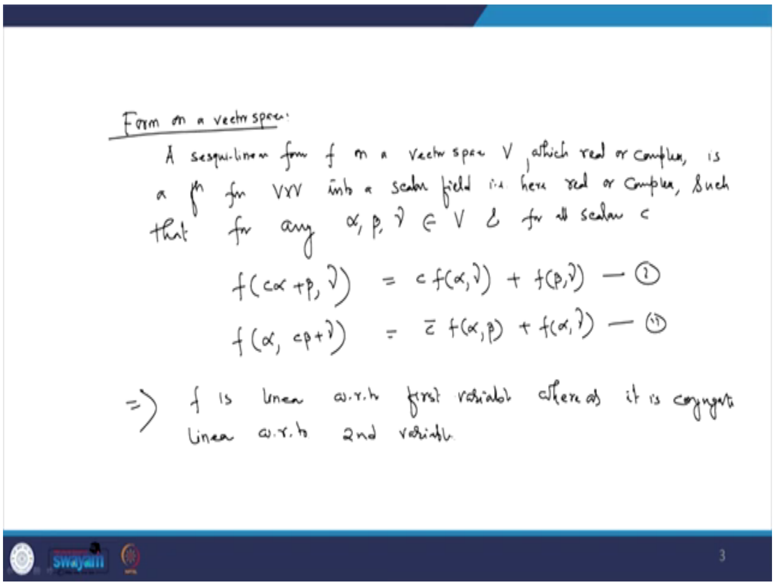

So, what is form on a vector space, this form I will say the sesqui-linear form. So, what is this sesqui-linear form on a vector space? Form on a vector space: = A sesqui-linear form f on a vector space V which is real or complex, I mean this vector space is real or complex vector space. So, is a function from VxV into a scalar field that is here real or complex such that for any $\alpha, \beta, \gamma \in$ V

& for all scalar c may be real complex such that f(c$\alpha + \beta, \gamma$) = cf(α, γ) + f(β, γ) → (i) & f$(\alpha$, $c\beta + \gamma)$ = $\bar{c}$ f(α, β) + f(α, γ) → (ii) ⟹ f is linear with respect to first variable where as it is conjugate linear with respect to second variable. That is why we are saying it is sesqui-linear form this general definition of the form on vector space.

I have not mentioned whether vector space is an inner product space or not. I am simply saying that if you consider a vector space over a real or complex field. Sesqui-linear form is defined like this.

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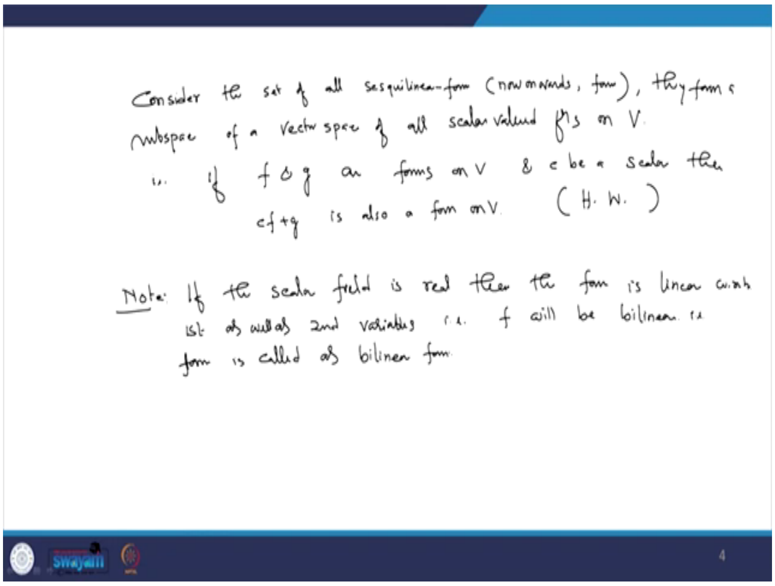

In fact, if we consider the set of all sesqui-linear form now or not only I will say forms as subspace. They form a subspace of a vector space of all scalar value functions on V, I mean to say if f & g are forms on V and c be a scalar. Then we can check (cf + g) is also a form on V. This part you can quickly check as in home assignment, so, I can give you the homework you can check it.

Another interesting fact say if the scalars are real. If the scalar field, it is real then the form is linear with respect to first, as well as second variables. That is f will be bilinear I mean in that case the form is called as bilinear form. So, this is the general definition of forms over a vector space, but our interest is to understand the form over a finite-dimensional inner product space.

Specifically, our objective is to understand whether, if I consider the subspace of all forms on vector space which is inner product vector space V. We want to know, is there any correlations between the set of all forms of a vector space and set up all linear operator on the vector space?

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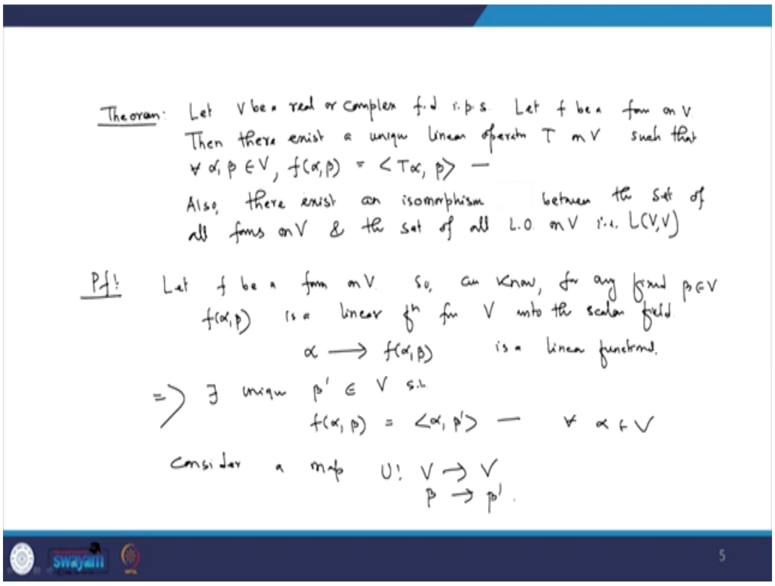

Now, I will basically show that there exist for each form on an inner product space. There exists a linear operator on that inner product space which are related by if I consider the form is f on inner product there will be a linear operator T on V such that f(α, β) = $< T\alpha, \beta >$ ∀ $\alpha, \beta \in$ V & T(f) will be unique. So, this is our now aim to establish these results.

Now, this type of results holds good also even in form normal space. But I will limit here over the finite-dimensional space only. So, this one I will prove in terms of these terms it is like this. Let V be a real or complex finite-dimensional inner product space. Let f be a form on V. Then there exist a unique linear operator T on V such that ∀ $\alpha, \beta \in$ V, f(α, β) = $< T\alpha, \beta >$

Also, there exists an isomorphism between the set of all forms on V and the set of all linear operator on V i.e. L(V,V). So, f to T(f) is a basically 1-1 & onto linear transformations between the space of all forms on vector space and set of linear operators on V i.e. L(V,V). So, let us quickly see the proof of this results. So, let f be a form on V. So, we know from the definition of the form for any fixed $\beta \in$ V, f(α, β) is a linear function from V into the scalar field.

To the corresponding scalar field, I mean it is a linear function that is $\alpha \rightarrow$ f(α, β) for any fixed β is linear, say, linear functional $\Longrightarrow \exists\ \beta' \in$ V such that f(α, β) = $< \alpha, \beta' >$ ∀ $\alpha, \beta \in$ V because we know from our previous results for any linear functionals there exist a unique element β'.

And being such that linear functional can be written as $< \alpha, \beta' >$ ∀ $\alpha, \beta \in$ V. So now, consider map of U: V → V which defined $\beta \rightarrow \beta'$.

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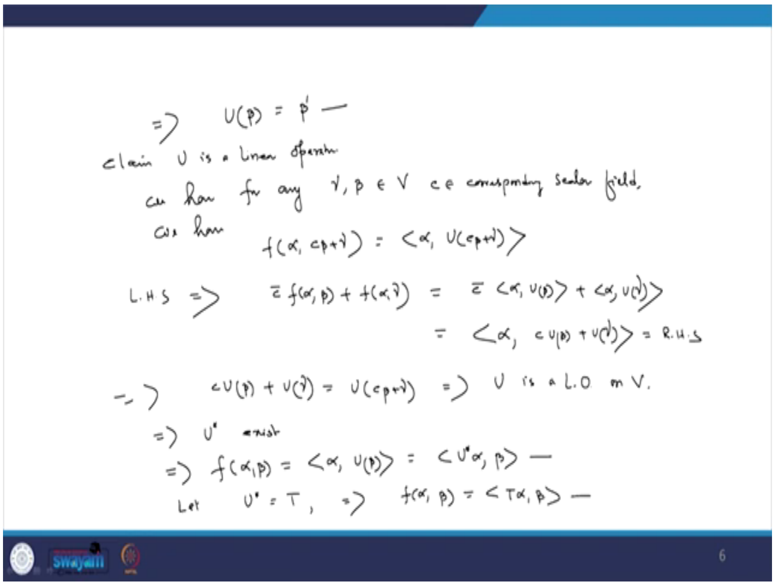

This implies $\mathrm{U}(\beta) = \beta'$. Claim U is a linear operator. We have for any $\gamma, \beta \in$ V & c $\in$ corresponding scalar field. We have according to the definitions $\mathrm{f}(\alpha, c\,\beta + \gamma) = <\alpha, U(c\beta + \gamma)$ & L.H.S. $\Longrightarrow$ $\bar{c}\,\mathrm{f}(\alpha,\beta) + f(\alpha,\gamma) = \bar{c} < \alpha, U(\beta) > + < \alpha, U(\gamma) > = < \alpha, cU(\beta) + U(\gamma) >$ = R.H.S. $\Longrightarrow$ $c\,\mathrm{U}(\alpha) + U(\gamma) = U(c\beta + \gamma)$

$\Longrightarrow$ U is a linear operator on V and since U is a linear operator on a finite-dimensional inner product space so, U^* exist.

$\Longrightarrow U^*$ exist I mean adjoint of U also exist $\Longrightarrow \mathrm{f}(\alpha,\beta) = < \alpha, U(\beta) > = < U^*\alpha, \beta >$. Let $U^* =$ T $\Longrightarrow \mathrm{f}(\alpha,\beta) = < T\alpha, \beta >$ claim T is unique.

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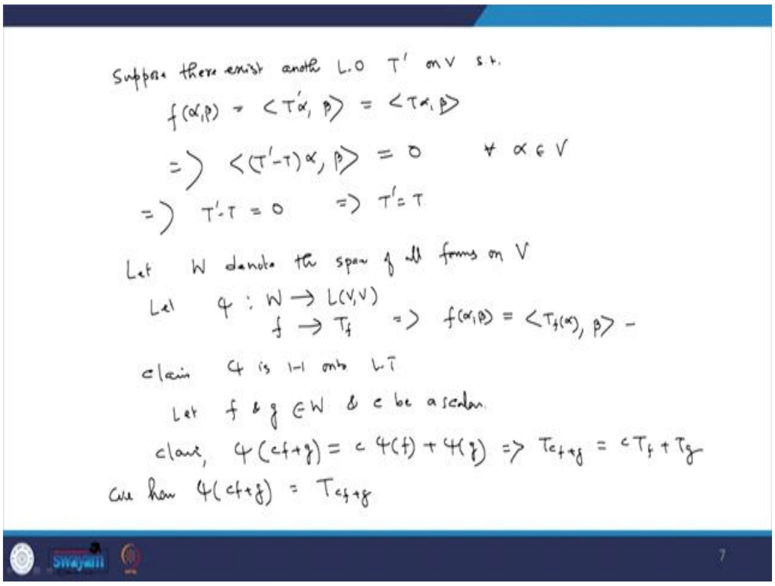

Suppose there exist another linear operator T' on V such that f$(\alpha, \beta) = < T'\alpha, \beta >$ = $< T\alpha, \beta >$ $\Longrightarrow$ $< (T' - T)\alpha, \beta >$ = 0 $\forall$ $\alpha \in V \Longrightarrow (T' - T) = 0$ $\Longrightarrow T' = T$ because both T & T' and linear operator. So, I can use the property of the linear operator and based on that I can write this.

So, we have shown there exist a unique linear operator T associated to the form f such that f(α, β) = = $< T\alpha, \beta >$. Now, our claim is that the space of all forms on the inner product space V must be isomorphic to L(V, V). So, let W denote the space of all forms on V.

Let φ: W $\rightarrow$ L(V, V) defined by f $\rightarrow$ T_f $\Longrightarrow$ $f(\alpha, \beta) = < T_f\alpha, \beta >$ claim φ is a 1-1 & onto linear transformations. Let f & g $\in$ W & c be n scalar. Claim φ(cf + g) = cφ(f) + φ(g) $\Longrightarrow$ $T_{(\mathrm{cf + g})}$ = cT_{f} +T_{g} we have φ(cf + g) = $T_{(\mathrm{cf + g})}$ = cT_{f} +T_{g}

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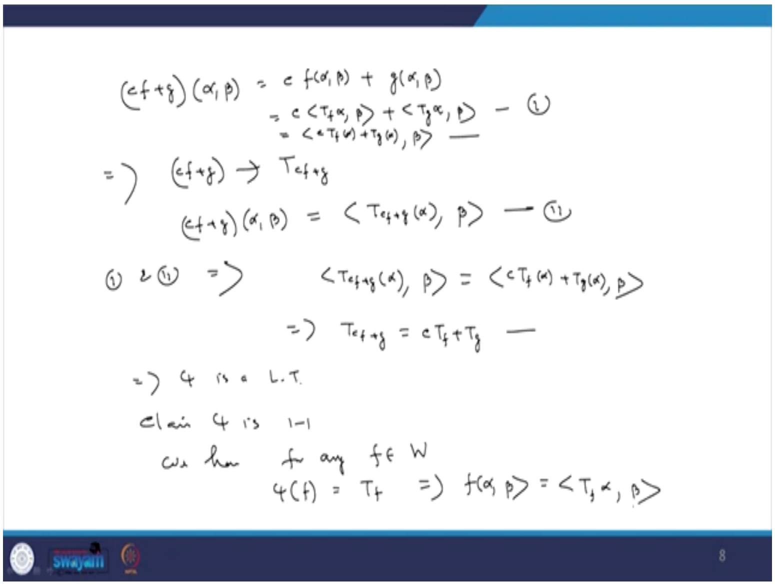

Now, (cf + g) (α, β) = cf(α, β) + g(α, β) = c $< T_f\alpha, \beta >$ + $< T_g\alpha, \beta >$ = $< c\, T_f(\alpha) + T_g(\alpha), \beta >$ where c $< T_f, \beta > + < T_f\alpha, \beta >$ $\rightarrow$ equation (i) $\Longrightarrow$ If (cf + g) $\rightarrow$ $T_{(cf+g)}$

$\Longrightarrow$ (cf + g) (α, β) = $< T_{cf+g}(\alpha), \beta >$ $\rightarrow$ equation (ii) $\Longrightarrow$ from (i) & (ii) $\Longrightarrow$ $< T_{cf+g}(\alpha), \beta >$ $= < c\, T_f(\alpha) + T_g(\alpha), \beta > \Longrightarrow T_{cf+g} = cT_f + T_g$

$\Longrightarrow$ φ is a linear transformation. Now, claim is φ is 1-1 we have for any f $\in$ W such that φ(f) = T_f $\Longrightarrow$ f(α, β) = $< T_f(\alpha), \beta >$

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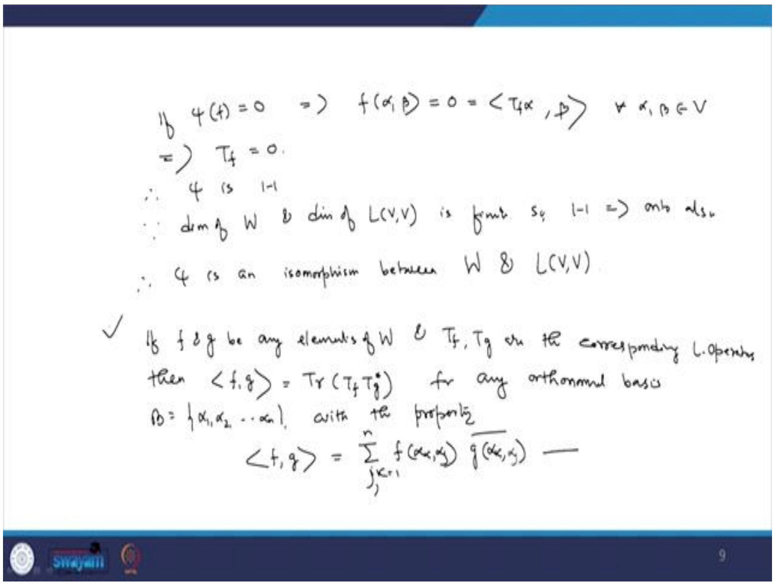

If φ(f) = 0 $\Longrightarrow$ f(α, β) = 0 = < $T_f(\alpha), \beta$ > $\forall$ $\alpha, \beta \in$ V $\Longrightarrow$ T_f = 0, so, φ is 1-1. Since, we is defined over two finite-dimensional vector space. So, therefore, φ also onto since dimension of W & dimension of L(V,V) is finite. So, 1-1 $\Longrightarrow$ onto also

So, φ is an isomorphism between W and L(V,V). So, you see that for each form on a finite-dimensional inner product space, there exist a unique linear operator T such that $f(\alpha, \beta) = < T\alpha, \beta >$. In fact, due to this isomorphism between the W and L(V, V), we can also say one nice results it is like this if f & g $\in$ W & T_f, T_g are the corresponding linear operators.

Then one can introduce an inner product on this space W then, < f, g > = $T_r\ (T_f, T_g{}^*)$ for any orthonormal basis B = $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$ with the property that < f, g > = $\sum_{j,k=1}^{n} f\left(\alpha_k, \alpha_j\right)\ \overline{g(\alpha_k, \alpha_j)}$

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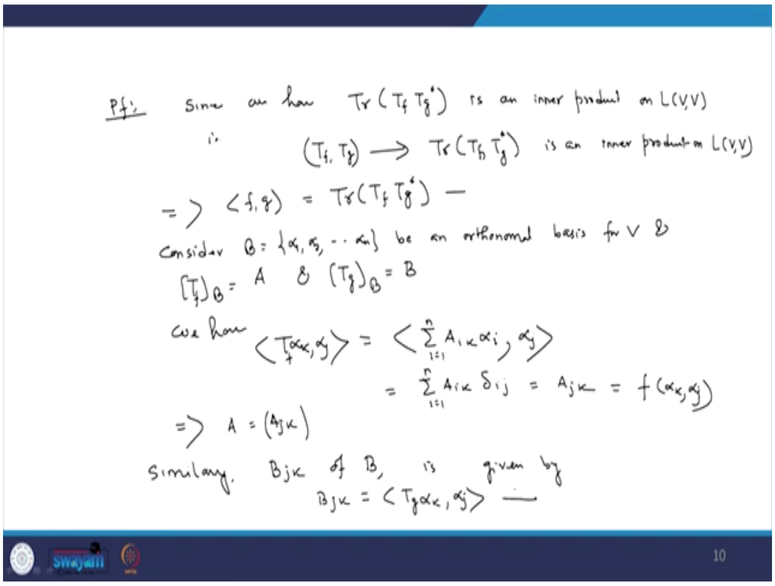

The proof of this result is also very straightforward. Since we know, T_r $(T_f, {T_g}^*)$ is an inner product inner product on L(V, V). I mean to say that is the functions (T_f, T_g) $\rightarrow$ T_r $(T_f, {T_g}^*)$ is an inner product on L(V, V) because inner product is also a function from L(V, V) to some scalar value complex number so, like this.

This implies that < f, g > = T_r $(T_f, {T_g}')$ because there is a one to one relation between the set of all forms on V and to L(V, V). I consider let B = $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an orthonormal basis for V and $[T_f]_B$ = A & $[T_g]_B$ = B.

Then we have $< T_f\alpha_k, \alpha_j > = < \sum_{i=1}^{n} A_{ik}\, \alpha_i, \alpha_j >$ = $\sum_{i=1}^{n} A_{ik}\, \delta_{ij}$ = A_{jk} = f(α_k, α_j) $\Longrightarrow$ A = (A_{jk}) Similarly, B_{jk} of matrix B is given by B_{jk} $= < T_g\alpha_k, \alpha_j >$

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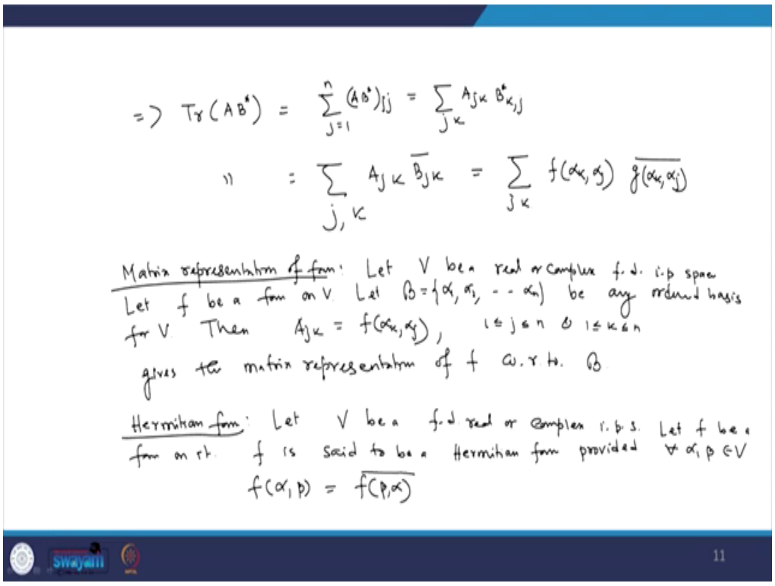

$\Longrightarrow$ Tr(AB^*) = $\sum_{j=1}^{n} (AB^*)_{ij} = \sum_{i,j=1}^{n} A_{jk}\, {B^*}_{k,j} \Longrightarrow$ Tr(AB^*) = $\sum_{j,k}^{n} A_{jk}\, \overline{B_{jk}}$ = $\sum_{j,k=1}^{n} f\big(\alpha_k, \alpha_j\big)\ \overline{g(\alpha_k, \alpha_j)}$. Now, let us see whether we can define a matrix representation of the form or not.

As we know, the operator T defined over a finite-dimensional vector space one can have a matrix representation of the operator with respect to a given order basis. So, similar concept also can be introduced for the forms. So, matrix representation of form but this is basically valid only for finite dimensional space, let V be a real or complex finite dimensional inner product space.

Let f be a form on V, let B = $\{\alpha_1, \alpha_2, \dots\ \alpha_n\}$ be any order basis for V. So, I am not taken here this basis as an orthonormal basis I have taken simply an order basis. Then A_{jk} = $f\big(\alpha_k, \alpha_j\big)$. So, if I define like this A_{jk} equal to like this for 1 $\le j \le$ n & 1 $\le k \le$ n gives the matrix representation of f with respect to order basis B.

The matrix representation of a form may not be same. The matrix representation of the corresponding linear operator they will be equal if we consider order basis as an orthonormal basis, otherwise the matrix representation of form will be different from the matrix representation of the corresponding operator. Let me define another terminology called Hermitian form.

Hermitian Form: - Let V be a finite-dimensional real or complex in our product space. Let f be a form on it f is said to be a Hermitian form provided $\forall\ \alpha, \beta \in$ V, $f(\alpha, \beta) = \overline{\mathrm{f}(\beta, \alpha)}$

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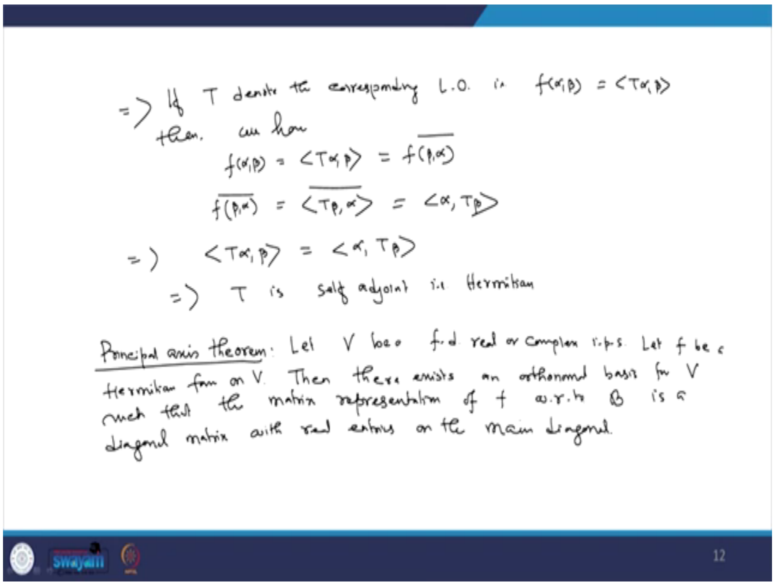

$\Longrightarrow$ If T denote the corresponding linear operator i.e. $f(\alpha, \beta) = <\mathrm{T}\alpha, \beta>$ then we have $f(\alpha, \beta) = <\mathrm{T}\alpha, \beta> = \overline{\mathrm{f}(\beta, \alpha)}$ & $\overline{\mathrm{f}(\beta, \alpha)} = <\overline{\mathrm{T}\beta, \alpha}> = <\alpha, T\beta> \Longrightarrow <\mathrm{T}\alpha, \beta> = <\alpha, T\beta> \Longrightarrow$ T is self-adjoint that is Hermitian.

So, if the form is Hermitian then associated operator is also Hermitian. So, this, given nice clue to establish a nice result that is called principal axis theorems. What is that? Principle Axis Theorem: - Let V be a finite-dimensional real or complex inner product space. Let f be a Hermitian form on V. So, let V be a final-dimensional real or complex inner product space. Let f be a Hermitian form on V.

Then, there exists an orthonormal basis for the space V such that the matrix representation of f with respect to B is a diagonal matrix now matrix with real entries on the main diagonal.

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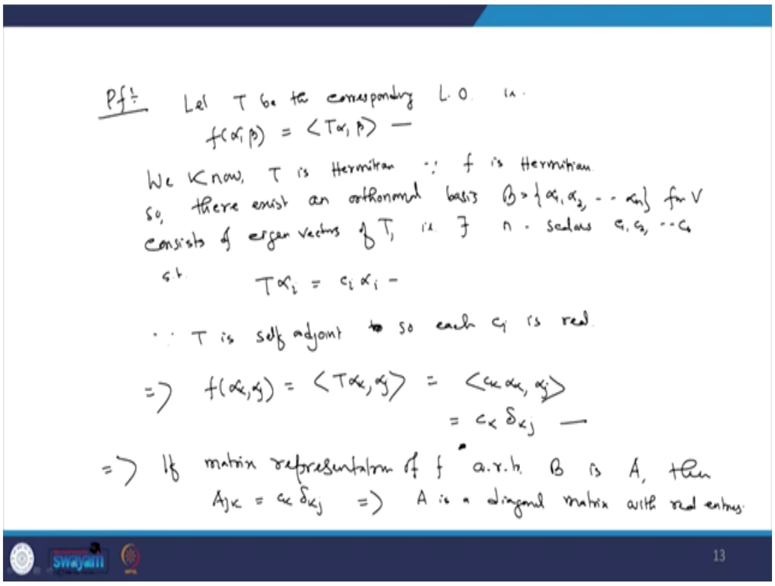

The proof of this result is basically application of the, what we have already learned. Proof: - Let T be the corresponding linear operator i.e. $f(\alpha, \beta) = < T\alpha, \beta >$, We know T is Hermitian, since f is Hermitian, so, there exists and orthonormal basis B = $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ for V consists of eigenvectors of T, i.e. there exists n scalars $e_1, e_2, \dots, e_n$ such that $T\alpha_i = c_i\alpha_i$

Since T is self-adjoint each c_i is real. Already we have proved because self-adjoint operator having eigenvalues are real. So, this implies, if I consider the matrix representation of the form A with respect to this order basis. I have f$(\alpha_k, \alpha_j > = < T\alpha_k, \alpha_j > = < c_k\alpha_k, \alpha_j > = c_k\delta_{kj}$

Since $\alpha_1, \alpha_2, \dots \alpha_n$ this is a orthonormal basis so, it will be basically like this. This δ_{kj} is a chronicle delta that is equal to 1 if k = j. So, this implies, if matrix representation of f with respect to order basis B is A then $A_{jk} = c_k\delta_{kj}$ so, this implies A is a diagonal matrix with real entry.

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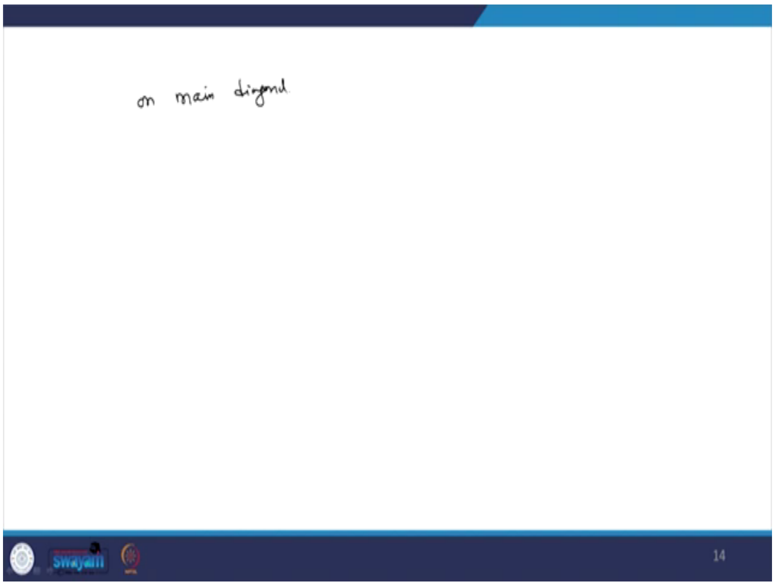

This is a nice result which will be utilized to solve many problems in terms of quadratic forms, positive forms in other different forms. However, due to time limitations, I could not talk many more interesting results. But I have initiated what I promised in my promo. I hope this initiation will help you to face the all the endeavours related to linear algebra problems, all the problems due to linear algebra.

When you face in your different areas. I am sure this will help you and boost you. All the best.