

Advanced Linear Algebra
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Lecture – 6
Basis and Dimension Part - 1

Welcome to today's lecture. In our last lectures, we have introduced the concept of vector space, subspace and we have seen different types of subspaces or different vector spaces and their characteristics also. We have defined a linear combination of vectors and we also introduced a subspace which is basically a collection of all possible linear combinations of a subset of a vector space also. Now, let me slightly modify and introduce another terminology that is called linear span of a subset of a vector space.

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Linear span of S:- Let V be a vector space over a field F .
 Let S be a subset of V . The linear span of S is set of all linear combination (LC) of elements of finite sets of elements of S . Let us denote L.S. of S as $L(S)$.
 $L(S)$ is also a subspace of V .
 \therefore for any $\alpha \in L(S)$ & $\beta \in L(S)$
 $\alpha = \sum_{i=1}^m c_i \alpha_i$ & $\beta = \sum_{j=1}^n t_j \alpha_j$, where $\alpha_i \in S$ & $t_i, c_j \in F$
 $\alpha = \sum_{i=1}^m c_i \alpha_i$ & $\beta = \sum_{j=1}^n t_j \alpha_j$
 let c be any other constant from F
 $c\alpha + \beta = c \sum_{i=1}^m c_i \alpha_i + \sum_{j=1}^n t_j \alpha_j = \sum_{i=1}^m (c c_i + t_i) \alpha_i + \sum_{j=m+1}^n t_j \alpha_j$ if $m < n$
 $\Rightarrow c\alpha + \beta \in L(S)$.

The linear span of a subspace of a vector space is basically if you consider a vector space say V over the field F , let S be a subset of V , then the linear span of S is a set of all linear combinations of elements of finite sets of elements of S . So, that is basically linear span of S . So, this will be also a subspace. So, let us denote a linear span I can denote as an LS linear span of say S as $L(S)$.

Since it is a collection of all linear combinations of elements of finite sets of elements of S , so similar to my previous proof related to V that collection of all the possible linear combinations of elements of a subset of vector space as a subspace, so here also it can easily so that $L(S)$ is also a

subspace of V . Since the reason is for any say $\alpha \in L(s)$ and $\beta \in L(s)$, so let us consider $\alpha = \sum_{i=1}^m c_i * \alpha_i$, and $\beta = \sum_{j=1}^n t_j * \alpha_j$. Where your $\alpha_i \in S$ & t_j and $c_i \in F$. Since I have already told that I do not want to write star frequently, so I simply write down $\alpha = \sum_{i=1}^m c_i * \alpha_i$, and $\beta = \sum_{j=1}^n t_j * \alpha_j$. Now, let c be any other constant from F , then we have $(c \alpha + \beta) = c \sum_{i=1}^m c_i * \alpha_i + \sum_{j=1}^n t_j * \alpha_j$.

So, this is again a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_j$. If $m > n$, then I can write down in the proper form that it is a LC of linear combination of α_i also, $\sum_{i=1}^m c(c_i + t_i) \alpha_i + \sum_{j=m+1}^n t_j \alpha_j$ if $m < n$. So, this implies that $(c \alpha + \beta) \in L(s)$, so implies it is subspace of vector space also. So, linear span of s is also subspace of the vector space V .

We are basically interested to generalize the concept of vector space by you know we want to also add a concept of dimension to the vector space also. Now, to assign a dimension concept to the vector space let me introduce some more terminology.

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Linearly dependent set: Let V be a vector space over a field F . Let S be a subset of V . Then S is said to be a linearly dependent subset of V provided there are say n distinct elements $\alpha_1, \alpha_2, \dots, \alpha_n$ of S & n scalars c_1, c_2, \dots, c_n such that

$$c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_n \alpha_n = 0$$

where not all c_i are equal to zero.

Else, S is said to be a linearly independent subset of V .

Ex $V = F^2$. Consider $S = \left\{ \begin{matrix} \alpha_1 \\ (1, 0) \end{matrix}, \begin{matrix} \alpha_2 \\ (0, 1) \end{matrix} \right\}$

$$c_1 \alpha_1 + c_2 \alpha_2 = 0$$

$$\Rightarrow c_1 (1, 0) + c_2 (0, 1) = (c_1, c_2) = (0, 0)$$

$$\Rightarrow c_1 = 0, c_2 = 0 \text{ is the only soln } \therefore S \text{ is a linearly independent subset of } V.$$

The first is linearly dependent set. Let V be a vector space over a field say F . Let S be a subset of V . Then S is said to be a linearly dependent subset or set of V provided there are say n distinct elements $(\alpha_1, \alpha_2, \dots, \alpha_j)$ of S and n scalars (c_1, c_2, \dots, c_n) such that $c_1 \alpha_1 + c_2 \alpha_2 + \dots + c_n \alpha_n = 0$ where not all $c_i = 0$. I mean to say there are some c_i which are also nonzero, but this relation holds good.

So, then I will say that the set S which is subset of vector space V will be linearly independent subset of V or we can say linearly dependent set of V . Otherwise S is said to be linearly independent subset set of V . So, let us take some examples. So, let me consider a very simple space say $V = F^2$, I mean set of space of all two tuples over the F . So, we know this is a vector space.

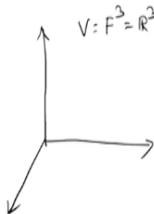
And with respect to the standard vector addition and scalar multiplication, I have already defined what do you mean by standard vector addition for F^n and standard scalar multiplication for F^n also. So based on that V is a vector space. So, consider $S = \{(1,0), (0,1)\}$. So, I have check whether r this is linearly independent or not, Now, we see for any linear combination we take, suppose the linearly dependent, then there will be some c_1, c_2 .

Such that suppose this is I have given as α_1, α_2 , then I will have basically $c_1 \alpha_1 + c_2 \alpha_2 = 0$, where all c_i not equal to 0, means at least some c_i will be nonzero. Let me check it. So, this means that I will have $c_1(1,0) + c_2(0,1) = (c_1, c_2) = (0,0)$ means $(0,0)$ is a vector coordinate. So, this implies that these two vectors are equal if their corresponding component are equal.

So this implies that $c_1 = 0, c_2 = 0$ are only solutions, so your capital S is a linearly independent subset of V . So, note that the LI, LI stands for linearly independent.

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Ex-2 $V = \mathbb{R}^3$
 $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} \right\}$
 check whether B is L.I or L.D.
 we have $1\alpha_1 + 1\alpha_2 - \alpha_3 = 0$
 $\Rightarrow B$ is a L.D. set of V .



Ex. $V = F^n$
 $B = \{ e_1 = (1, 0, \dots, 0), e_2 = (0, 1, \dots, 0), \dots, e_i = (0, \dots, 1, \dots, 0), \dots, e_n = (0, \dots, 0, 1) \}$
 $\therefore B \subset V$
 for any $\alpha \in V$
 $\alpha = (x_1, x_2, \dots, x_n)$
 \Rightarrow for any L.C. of e_1, e_2, \dots, e_n
 $x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n = (x_1, x_2, \dots, x_n) = 0$
 \Rightarrow each $x_i = 0 \therefore B$ is L.I. subset of V .

So let me take some another example. So, in my Euclidean space, this is basically the space equal to say F^3 which is I am writing as \mathbb{R}^3 , I mean $F =$ capital \mathbb{R} real line. So we know that $V = \mathbb{R}^n$ is also vector space with respect to standard vector addition and scalar multiplication over the

real line \mathbb{R} . Now $V = \mathbb{R}^3$ and $B = (1, 1, 0), (1, 0, 0), (2, 1, 1)$. Let me quickly check whether this set is linearly independent or dependent, B is linearly independent or linearly dependent.

So, suppose if I give the name this is basically say $\alpha_1, \alpha_2, \alpha_3$, if I give the name like this, we have here $1 \cdot \alpha_1 + 1 \cdot \alpha_2 - \alpha_3 = 0$ Because if I add these two vectors α_1, α_2 , I am getting basically α_3 . So, this implies that B is a linearly dependent set of V . So, one can also extend this instead of \mathbb{R}^3 if I consider $V = F^n$.

And if consider $B = \{e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \text{ and } e_i = (0, 0, \dots, 1, 0, \dots, 0), e_n = (0, 0, \dots, 0, 1)\}$. So, B is a subset of V , now we can immediately check that B is linearly independent set of V because for any $\alpha \in B$, α will be of the form of what? $\alpha = (x_1, x_2, \dots, x_n)$. So, it is n -tuple space.

So, this implies one can write down for any linear combinations of e_1, e_2, \dots, e_n , $x_1\alpha_1 + x_2\alpha_2 + \dots + x_n\alpha_n = (x_1, x_2, \dots, x_n) = 0$, this implies that each $x_i = 0$, so B is a linearly independent subset of V . So, we have understood what is the linearly independent and linearly dependent.

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Basis: Let V be a vector space over F . A subset B of V is said to be basis of V provided V is spanned by B & B is a L.I. subset of V .

Ex $V = F^n$
 We know $B = \{e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_i = (0, 0, \dots, 1, 0, \dots, 0), e_n = (0, 0, \dots, 0, 1)\}$

We know any element of V , say α , is of the form of $\alpha = (x_1, x_2, \dots, x_n)$
 $\alpha = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$
 $\Rightarrow \text{span}(B) = V$
 Already seen, B is a L.I. subset of F^n
 $\Rightarrow B$ is a basis of V , In fact, it is called an standard basis in F^n

Now let me introduce another terminology called the basis. Let V be a vector space over F . A subset B of V is said to be the basis of V provided V is spanned by B and B is a linearly independent LI set or subset of V . So a subset of a vector space will be basis for the corresponding space provided it span the space, I mean to say linear span of B equal to B and L

of B equal to V and the B is a linearly independent subset of V.

So for example $V = F^n$. We know the collection $B = \{ e_1 = (1, 0, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 1, 0, \dots, 0), e_n = (0, 0, \dots, 0, 1) \}$. So, if I consider this is a subset of the vector space V, then I can immediately show that this will span your vector space V. We know any element of V say α is of the form of your $\alpha = (x_1, x_2, \dots, x_n)$. and $x_i \in F$.

Here I have taken capital F again real number, you may take all complex, no problem. Now, I see that $\alpha = x_1 e_1 + x_2 e_2 + \dots + x_n e_n$. So, this implies $L(B) = B$. Now only thing is I have to prove that it is linearly independent. Already seen B is a linearly independent subset of F^n . So, this implies B is a basis of V. In fact, we used to say this is called the standard basis. It is called as standard basis in a F^n .

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Ex Let $V =$ set of all polynomial f's of degree less than equal to n, over F
 consider $B = \{ 1, x, x^2, \dots, x^n \} \subset V$

For any $p \in V \Rightarrow p = p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n$
 $= p_0 \alpha_1 + p_1 \alpha_2 + \dots + p_n \alpha_{n+1}$

$\Rightarrow L(B) = V$

claim B is L.I subset of V
 Suppose not,
 $a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n = 0 \quad \forall x \in F$

$\therefore p(x) = \sum_{i=0}^n a_i x^i$ is a polynomial of degree n, so it has n roots i.e.
 p is zero on the line F if F is real number line at most at
 n distinct pts.
 $\therefore p(x) = 0 \quad \forall x \in F$ is not possible. $\therefore B$ is a basis of V.

Let V equal to set of all polynomial functions of degree less than equal to n and defined over the field capital F. We have also seen that capital V is a vector space also. Now, if I consider $B = \{ 1, x, x^2, \dots, x^n \}$, which is a subset of V. Now, claim this B is a basis for V. So, to prove this as a basis for V we have to show that it span B and it is also linearly independent. For any element $p \in V$ implies that $p = p_0 + p_1 x + p_2 x^2 + \dots + p_n x^n$.

This is of this form where $p_0, p_1, \dots, p_i \in F$. So, this implies that this is nothing p_0 , so if I give the name this is my α_1 , this is my α_2 and like this is my α_n , so basis B consistent basis $\alpha_1, \alpha_2, \dots, \alpha_n$ where $\alpha_1 = 1, \alpha_2 = x, \alpha_3 = x^2$ and $\alpha_n = x^n$. Then I am able to write $p = p_0 \alpha_1 + p_1 \alpha_2 + p_2 \alpha_3 + \dots$

$$\dots + p_n \alpha_{n+1}.$$

So, this implies that $L(B) = V$, I mean set of all polynomial function of degree less than or equal to n over F . Now, the claim B is a linearly independent subset of V . Suppose not, then I will have linear combinations of these elements such that that will be equal to 0 where at least some coefficient will be nonzero. So, let me consider say linear combinations $a_0 1 + a_1 x + a_2 x^2 + \dots + a_n x^n$ such that this is true for all $x \in F$.

But friends as you know, this is a polynomial of degree n , so it will have only n roots, means it will be 0 only at n places over the F . So, it cannot be equal to 0 for all F . Since $\sum_{i=0}^n a_i x^i$ is a polynomial of degree n , so it has n roots that is if I denote this is basically $p(x)$ you know this that is p is zero on the line F if I say that if F is real line, then the p is zero on the line at most at n distinct points.

So, $p(x) = 0$ for all $x \in F$ is not possible, so B is a basis of V . If you consider V be the set of all polynomial functions of finite degree over the F , then we will see that to span V we need a basis having infinite number of terms. So, let me define the concept of dimension.

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Dimension of a vector space: Let V be a vector space over F & B be a basis of it. If B has finite number of elements, then V is called a finite dimensional vector space (f.d.v.s), else it is called an infinite dimensional vector space.

Ex Let P be an $n \times n$ invertible matrix over F . Then all columns of P will be a basis of $V = F^n$.

PF: Let p_1, p_2, \dots, p_n denote the 1st, \dots , n th columns of P .

$$\text{Let } x_1 p_1 + x_2 p_2 + \dots + x_n p_n = 0 \text{ --- (1)}$$

$$\Rightarrow PX = 0, \text{ where } P = [p_1 \ p_2 \ \dots \ p_n] \text{ \& } X = [x_1 \ x_2 \ \dots \ x_n]^T$$

$$\because P^{-1}PX = 0 \Rightarrow X = 0 \text{ is the only soln}$$

$$\Rightarrow \text{Sol } B = \{p_1, \dots, p_n\} \text{ is a L.I.}$$

$$\text{Let } Y = (y_1, y_2, \dots, y_n)^T \\ \text{can } \exists X \text{ s.t. } PX = \sum_{i=1}^n x_i p_i = Y \therefore X = P^{-1}Y \therefore B \text{ spans } V.$$

So, dimension of a vector space. Let V be a vector space over F and B be a basis of it. If B has finite number of elements, then V is called a finite dimensional vector space, now onwards I will

use this abbreviation (f.d.v.s), else it is called as an infinite dimensional vector space. So, we have seen that according to this definition F^n over F is a finite dimensional vector space and dimension is n .

A set of polynomial functions of degree less than or equal to n is again a finite dimensional vector space of dimension $n+1$, but if I consider V be the case of all polynomial functions over F , then the dimension of this vector space will be infinite because no finite linearly independent subset of the corresponding V cannot span the corresponding space. So, let me take some more other example of the basis.

Let P be an $n \times n$ invertible matrix over F . Then all columns of P will be a basis of vector space $V = F^{n \times 1}$. So, this is what we know again a vector space V . Now, I have considered P be an invertible matrix, claim that the columns of P form a basis for this corresponding space. So, this is again very straightforward because first I have to show that it is linearly independent.

So, let me consider P_1, P_2, \dots, P_n denote a first to n th columns of P . Suppose, there is a linear combination of P_1, P_2, \dots, P_n which is equal to 0. Let $x_1 P_1 + x_2 P_2 + \dots + x_n P_n = 0$. Then I have to prove that $x_1 = x_2 = \dots = x_n = 0$ the only solution. See, this is equal to 0 this means that I can write down this is your simply $PX=0$, I meant to say where $P = [P_1, P_2, \dots, P_n]$ and $X = [x_1, x_2, \dots, x_n]^T$.

So this linear combinations of the P_1, P_2, \dots, P_n is nothing $PX=0$. Since P is invertible So, P inverse like this, so $P^{-1}PA = 0$. So, this implies $X = 0$ is the only solution. So, this implies that set $B = \{P_1, P_2, \dots, P_n\}$ is a linearly independent set. Now, I have to show that it also span $F^{n \times 1}$. So, let capital $Y = [y_1, y_2, \dots, y_n]^T$, so this is a column vector which belongs to V .

Claim, $\exists X$ such that $PX = x_1 P_1 + x_2 P_2 + \dots + x_n P_n$. So, this is equal to $\sum_{i=1}^n x_i P_i = Y$. Since $PX=Y$ I have to know what is X . Now, P is invertible, multiplying P inverse to both the sides I am getting $X = P^{-1} Y$. So, we can immediately find out the corresponding n constant (x_1, x_2, \dots, x_n) such that $x_1 P_1 + x_2 P_2 + \dots + x_n P_n = Y$. So, this means that the B span V also. So, I think we have given enough number of examples of the basis.

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P is a $m \times n$ matrix & $m < n$
 $\Rightarrow PX=0$ will have non-trivial solⁿ
 $\Rightarrow \exists X = (x_1, x_2, \dots, x_n)^T$ s.t. $X \neq 0$ but $PX=0$

$$\Rightarrow \sum_{i=1}^n x_i \alpha_i = \sum_{i=1}^m \left(\sum_{j=1}^n P_{ij} x_j \right) \beta_i$$

$$P_{11}x_1 + P_{12}x_2 + \dots + P_{1n}x_n = 0$$

$$\vdots$$

$$P_{m1}x_1 + P_{m2}x_2 + \dots + P_{mn}x_n = 0$$

$\Rightarrow \exists X \neq 0$, But:
 $\sum_{i=1}^n x_i \alpha_i = 0$
 $\therefore S$ is a L.D subset of V .

Since P is an $m \times n$ matrix and $m < n$, so this implies if I consider $PX=0$ will have a non-trivial solution. So this means that $\exists X = [x_1, x_2, \dots, x_n]^T$ such that $X \neq 0$ but $PX=0$. So, this implies that $\sum_{i=1}^n x_j \alpha_i = \sum_{j=1}^m \left(\sum_{i=1}^n P_{ij} x_j \right) \beta_i$. Now, if I consider that X , I will have $P_{11}x_1 + P_{12}x_2 + \dots + P_{1n}x_n = 0, \dots$

$P_{m1}x_1 + P_{m2}x_2 + \dots + P_{mn}x_n = 0$, So, I will have this value capital X , this is equal to 0. This means that I am getting $\exists X \neq 0$, but your $\sum_{i=1}^n x_j \alpha_i$. So this implies S is a linearly independent subset of V . So, for today it is alright up to this, we will continue in the next class. Thank you.