Advance Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 59
Singular Value Decomposition of a Matrix

Welcome to the lecture series in Advanced Linear Algebra, friends; we have already seen for a linear operator defined over finite dimension vector space that a square matrix is considered over a field. We have seen the matrix defined over the complex field. Then it has eigenvalue and eigenvector. So, whether this concept is possible to say something the similar concept for a rectangular matrix defined over the complex field or some field that the question.

So, we know the concept of eigenvalue, eigenvector for a square matrix. But what can you say about the rectangular matrix? To answer this question today will basically apply the concept of spectral theory to answer this question.

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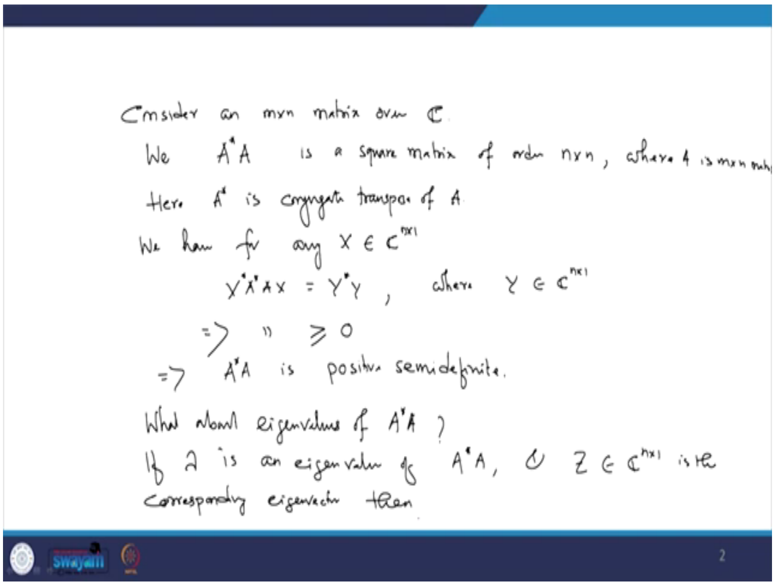

So, let me consider and mxn matrix over ℂ. We have A^*A is a square matrix of order nxn. Because here A^* is basically conjugate transpose of A to the notation I have used as a adjoint of A but we have defined adjoint. Basically, it is we are limited in this course the adjacent basically mean for operator or square matrix.

But here, if I use the adjoint for a rectangular matrix, this is, I am talking only simply conjugate transpose of this matrix A. So, here A^* is conjugate transpose of A. So that is why, if A is a mxn matrix. So, now we see, that we have for any X $\in \mathbb{C}^{nx1}$ the set of all the column vectors defined over the complex number.

Then you see that $X^*A^*AX = Y^*Y$, where Y $\in \mathbb{C}^{nx1}$ $\Longrightarrow$ $Y^*Y \geq 0 \Longrightarrow$ A^*A is positive semi-definite if it is strictly greater than 0 I could say that this is positive definite anyhow. So, here since I have picked up any randomly any matrix nxn matrix over the field $\mathbb{C}$.

So, I will say that this is a simply positive semi-definite. So, if it is positive semi-definite, then what can you say about the eigenvalues of this matrix? Now, what about the eigenvalues of A^*A is a question. See if λ is an eigenvalue of A^*A & Z $\in \mathbb{C}^{nx1}$ is the corresponding eigenvector.

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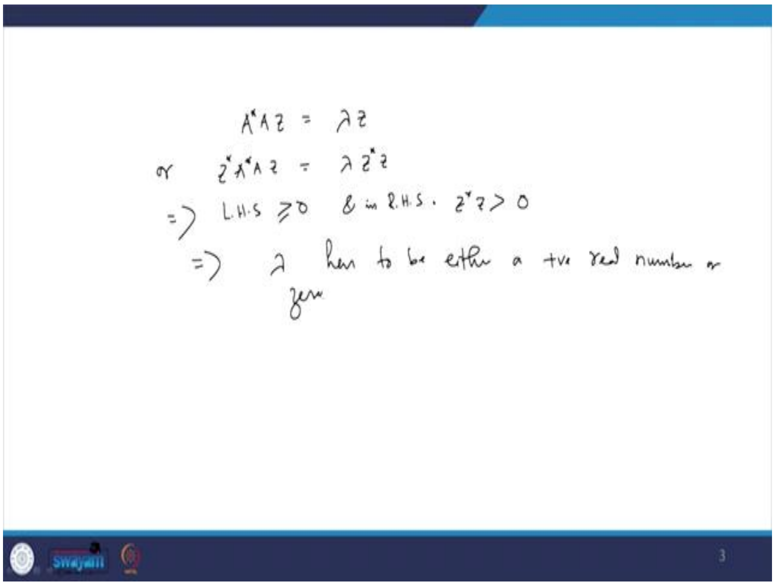

Then $A^*AZ = \lambda Z$ or $Z^*A^*AZ = \lambda Z^*Z$ $\Longrightarrow$ L.H.S. ≥ 0 & in RHS $Z^*Z > 0$ because Z is eigenvector, so, it has to be non-zero vector.

$\Longrightarrow \lambda$ has to be either if positive real number or 0. So, eigenvalues of A^* is either positive real number or 0.

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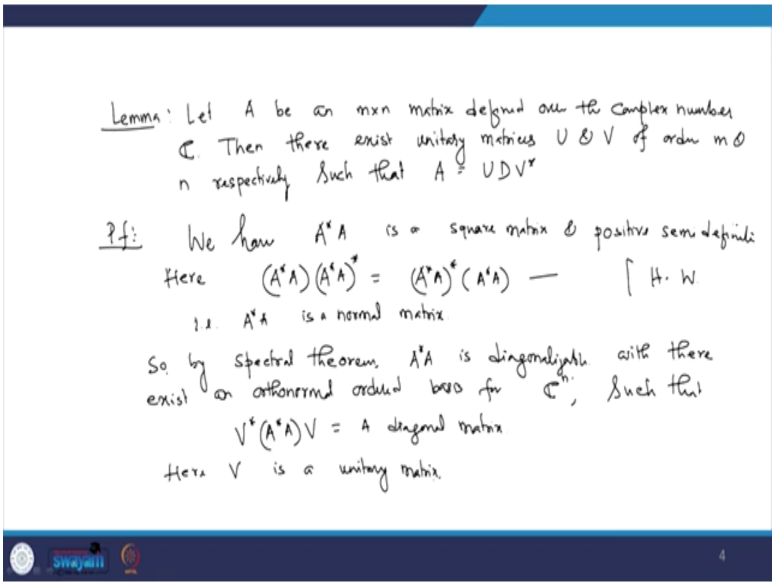

Now, based on this concept, let me write down one nice result that is in terms of Lemma is like this. Lemma:- Let A be an mxn matrix defined over the complex number ℂ. Then there exist unitary matrices U and V of order m and n respectively such that $A = UDV^*$ so, U and V are square matrices of order m and n respectively A is m xn matrix.

So, to prove this results I have to basically use the spectral theory. So, how we have to prove it let us see it. We have A^*A is a square matrix and positive semi-definite or I can say non-negative positive semi-definite. Also so, here $(A^*A)(A^*A)^* = (A^*A)^*(A^*A)$. You can quickly check that the matrix A^*A is a normal matrix.

So, you can check it this 1 asking you to do take care of the homework that show that A^*A is a normal matrix. So, by spectral theorems A^*A is diagonalizable with there exist an orthonormal order basis for $\mathbb{C}^n$ such that $V^*(A^*A)V = A$, it is diagonal matrix. Here V is a unitary matrix which columns are basically that orthonormal basis.

I mean we have already checked that if we consider normal matrix then there exists a unitary matrix V such that V^* into that normal matrix into V equal to diagonal matrix this diagonal matrix is consists of eigenvalues in the diagonal.

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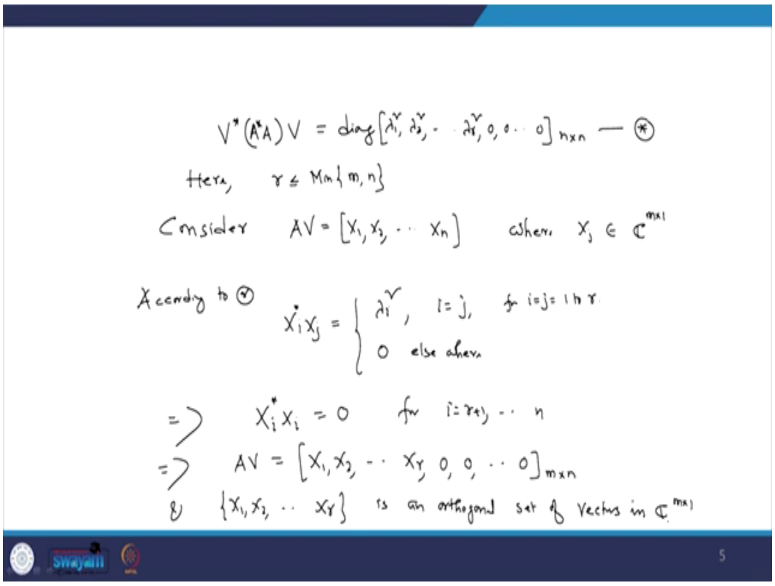

Here , since I have already shown eigenvalue of (A*A) is either positive real number or 0. So, if I consider there are exactly r number of non-zero eigenvalues and then I can write, $\mathrm{V}^*(\mathrm{A}^*\mathrm{A})\mathrm{V} = \mathrm{diag}[\lambda_1{}^2, \lambda_2{}^2, \ldots\ \lambda_r{}^2, 0, 0, \ldots 0] \rightarrow (*)$ so, this is a nxn matrix, where there are we have assumed exactly r number of eigenvalues $\lambda_1{}^2, \lambda_2{}^2, \ldots \lambda_r{}^2$ which are non-zero and rest are zero.

It is also true that your this r $\leq$ min{ m, n }. I mean the order of the matrix original matrix A. So, this results I have not proved that this number of non-zero Eigen value has to satisfy this relations. This is a you can consider as a assignment problem. Consider AV, since A is a m xn matrix and V is a nxn matrix. So, AV will equal which is basically mxn matrix.

Now, let me consider this is equal to the columns of the AV = $[X_1, X_2, \ldots\ X_n]$ where $X_i \in \mathbb{C}^{nx1}$. So, if you see the (*), according to the equation (*), we have $X_i{}^*X_j = \begin{cases}\lambda_i{}^T, & i = j,\ and\ i,j = 1\ to\ r \\ 0 & also\ above\end{cases}$, What I am going to say you, if I consider $X_i{}^*X_j$ because this is $\mathrm{V}^*(\mathrm{A}^*\mathrm{A})\mathrm{V}$.

I have taken the AV = $[X_1, X_2, \ldots\ X_n]$. Then AV* will be basically conjugate transfer this matrix only so, this means that I am getting $X_i{}^*X_j = \begin{cases}\lambda_i{}^T, & i = j,\ and\ i,j = 1\ to\ r \\ 0 & also\ above\end{cases}$. So, this implies that $X_i{}^*X_j = 0$, for i = r + 1 to n.

So, this implies your AV, I am getting the actual AV = [$X_1, X_2, \ldots X_r$, 0,0…0]. So, this is the mxn matrix like this and this set $X_1, X_2, \ldots X_r$ is an orthogonal set of vectors in $\mathbb{C}^{mx1}$.

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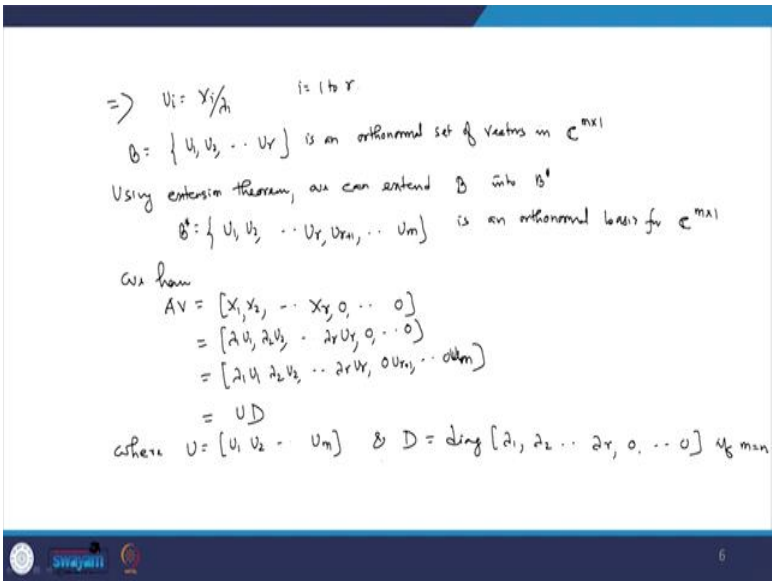

⟹ if I consider $U_i = \frac{X_i}{\lambda_1}$ for i = 1 to r. If I do like that then I am getting. B = [$U_1, U_2, \ldots U_r$] is an orthonormal set of vectors in $\mathbb{C}^{mx1}$. Since it is a orthonormal set of vector, so, definitely it will in an independent. So, therefore, I can use the extension theorems. So, using extension theorem we can extend B into B* by adding (m–r) more elements B* = {$U_1, U_2, \ldots U_r, U_{r+1}, \ldots U_m$}

is an orthonormal basis for $\mathbb{C}^{mx1}$. Now, let us rewrite we have, here AV = [$X_1, X_2, \ldots X_r$, 0,0…0] = [$\lambda_1 U_1, \lambda_1\, U_2, \ldots\, \lambda_r U_r$, 0,…0] = [$\lambda_1 U_1, \lambda_1\, U_2, \ldots\, \lambda_r U_r, 0U_{r+1}, \ldots 0U_m$] = UD ⟹ AV = UD where U = [$U_1, U_2, \ldots\, U_r$] & D = diag($\lambda_1 v_1, \lambda_2$, … λ_r, 0, … 0) if m = n.

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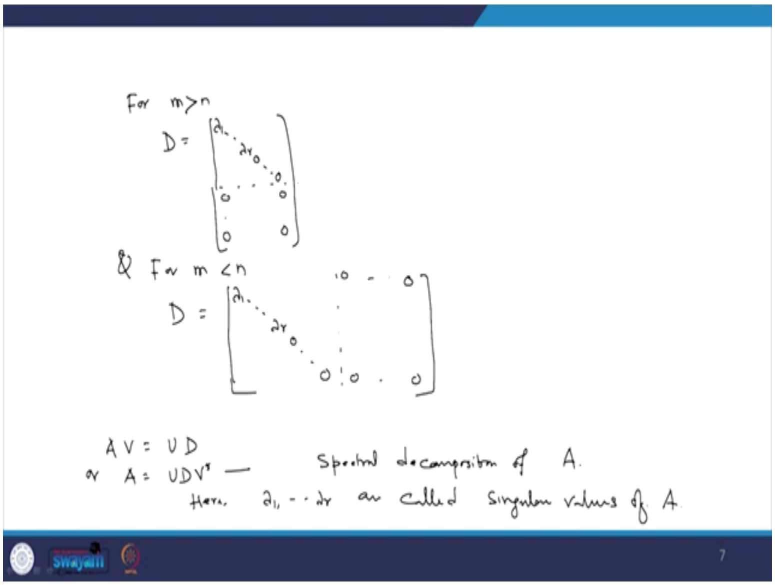

If m > n then $D = \begin{bmatrix} \begin{pmatrix} \lambda_1 & \cdots & \\ \vdots & \ddots & \vdots \\ & \cdots & \lambda_r \end{pmatrix} \\ \ldots\ldots\ldots\ldots\ldots. \\ 0 \ldots\ldots 0 \\ \\ 0 \ldots\ldots 0 \end{bmatrix}$ & for m < n, $D = \begin{bmatrix} \lambda_1 & \cdots & 0 & \cdots & 0 \\ \vdots & \lambda_r & \vdots\ \vdots & & \vdots \\ & \cdots & 0\ 0 & \cdots & 0 \end{bmatrix}$. So, this is called spectral decomposition of the matrix A.

Here we have AV = UD or A = UDV*, because V is unitary operator, so, inverse of it is basically is adjoint of it. So, A equal to this one this is called spectral decomposition of matrix A. Here $\lambda_1, \lambda_2, \ldots \lambda_r$ are called singular values of A. So, these are not eigenvalues of A. I am saying it is a singular values. This is a positive square root of the eigenvalues of A*A.

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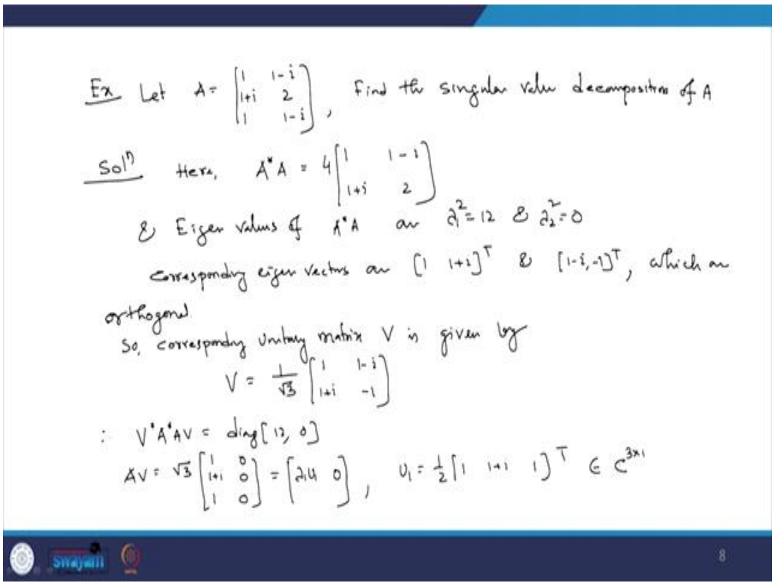

So now, let me consider examples and let me show you how to construct the matrix U and V for a given matrix A.

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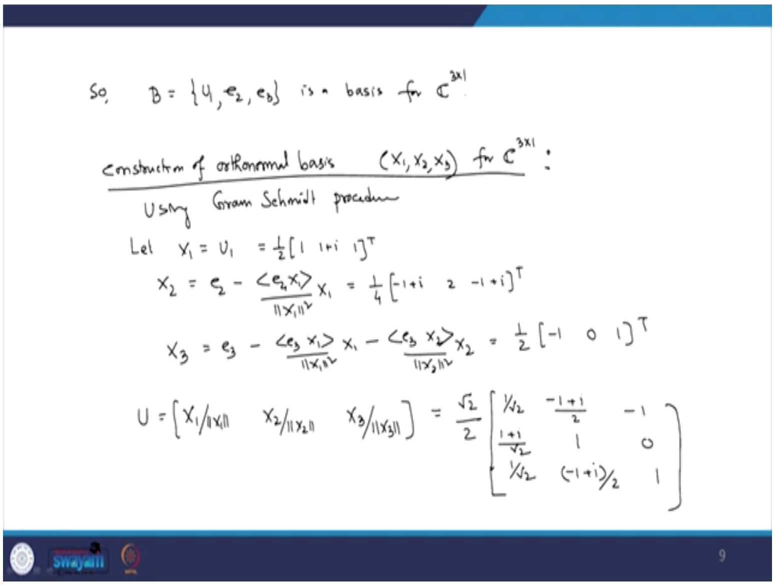

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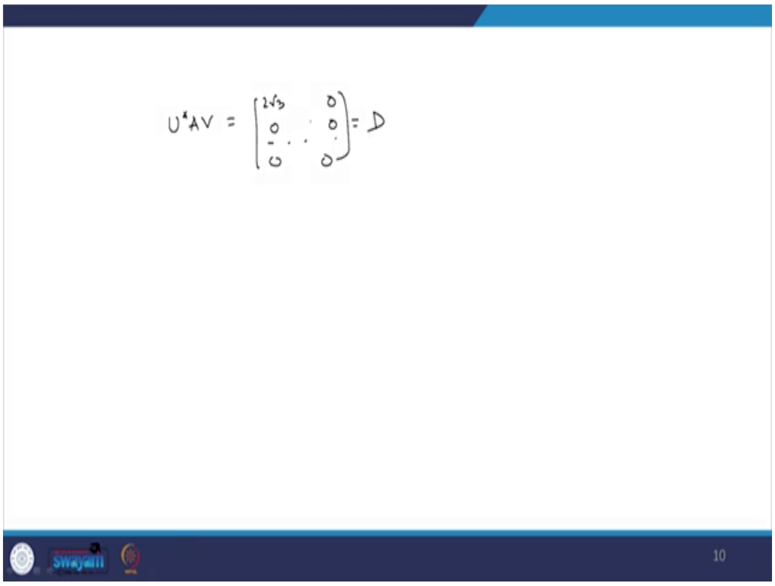

So, let us consider an examples and so that how to find the singular value decomposition of the matrix A. So, here I have taken, $A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \\ 1 & 1-i \end{bmatrix}$ matrix over the defined over the complex field C. I want to have a corresponding singular value decompositions. So, in the very beginning I have to check what is A^*A wil be 2x2 matrix, so $A^*A = 4\begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$.

Since it is square matrix, we can immediately find out the corresponding eigenvalues. And we can see the eigenvalues of the matrix A*A are ${\lambda_1}^2$ = 12 & ${\lambda_2}^2$ = 0.

So, we are looking basically, we are basically interested for the singular values of the matrix A in terms of λ_1 & λ_2. So, corresponding eigenvectors we have calculation can calculate that is $\{1,\ 1+i]^T$ and for the eigenvector associated to eigvalue 0 is $\{1-i,\ -1]^T$. So, these are also orthogonals. You can quickly check it that is correct, it is simply dot product or your standard inner product over the C^2, you can say it is equal to 0.

So, immediately we can find out the corresponding orthonormal by dividing the norm of these two columns. So, I can write down the corresponding unitary matrix V which is basically the consist of the orthonormal basis. The first element is eigenvector associated to 12 that is here $\frac{1}{\sqrt{13}}\begin{bmatrix} 1 \\ 1+i \end{bmatrix}$ & and the second one is $\frac{1}{\sqrt{13}}\begin{bmatrix} 1-i \\ -1 \end{bmatrix}$.

So, V*A*AV = diag{12, 0}. You can also cross check by simply multiplying this two matrices V*A* and AV. Now, let me consider if AV = $\sqrt{13}\begin{bmatrix} 1 & 0 \\ 1+i & 0 \\ 1 & 0 \end{bmatrix}$ = $[\lambda_1 U_1\ 0\ ]$

According to our analysis, I mean theorems, your AV must be equal to I mean if the first column of AV is the right hand side, $\lambda_1 U_1$ & second one is 0. So, let me consider $U_1 = \lambda_1 X_1$ where X_1 is basically first column of AV. So, here first column of AV is equal to $\sqrt{3}$ times this column. So, this divided by the corresponding eigenvalues. eigenvalues ah square root eigenvalue that is λ_1, $\lambda_1 = 2\sqrt{3}$.

So, if I do it then I am getting this ones. So, I have a U_1 and other column is 0. So, according to theorem, I can form a basis for the corresponding space C^{3x1}. By adding two more linear and independent element that I have taken from the standard basis on the C^{3x1} that is e_2 & e_3. What is e_2? $e_2 =$ (0 1 0), e_3 = (0 0 1). So, I have U_1, I have e_2, I have e_3

You can cross check this set of vector is a linearly independent fine. Now, I have to obtain the corresponding orthonormal basis for the space C^{3x1}. To get that once I have to use Gram-Schmidt orthogonal procedures. So, using that let me construct my first linear independent orthogonal element is $X_1 = U_1 = \frac{1}{2}\{1 \quad 1+i \quad 1]^T$ & $X_2 = e_2 - \frac{<e_1,X_1>}{||X_1||^2} X_1 = \frac{1}{4}\{-1+i \quad 2 \quad -1+i]^T$

$X_3 = e_3 - \frac{<e_2,X_1>}{||X_1||^2} X_1 - \frac{<e_3,X_2>}{||X_2||^2} X_2 = \frac{1}{2}[-1 \quad 0 \quad 1]^T$. Then I can again make it orthonormal by dividing by the norm of the corresponding vectors, column vectors. So that is, I can have a unitary matrix U = $\{\frac{X_1}{||X_1||} \quad \frac{X_2}{||X_2||} \quad \frac{X_3}{||X_3||}\} = \frac{\sqrt{2}}{2}\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1+i}{2} & -1 \\ \frac{1+i}{\sqrt{2}} & 1 & 0 \\ \frac{1}{\sqrt{2}} & \frac{-1+i}{2} & 1 \end{bmatrix}$ You see this matrix is a 3x3 square matrix.

So then, if I simplify that I am getting U*AV = $\begin{bmatrix} 2\sqrt{3} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \\ \ldots\ldots\ldots\ldots\ldots & & \\ 0 \ldots\ldots 0 & & \end{bmatrix}$ = D

Where U* will be again 3x3 matrix into AV, A is 3x2 matrix. So, U*A is basically the 3x2 matrix and V is again 2x2 matrix. So, it basically finally 3x2 matrix. So, I am getting that is a matrix of this structure. The first row is $2\sqrt{3}$ that is the singular value of the matrix A or positive square root of the first eigenvalues of A*A.

That is $2\sqrt{3}$ and second row consists of 0, 0 only and same to third row also 0, 0. So, this is my corresponding diagonal matrix. This is a rectangular diagonal matrix. So, in this way we have a singular value decomposition of the given matrix A. You can also do the same exercise for considering any matrix and do it have feelings about this decomposition and see the beauty of a spectral theorem.

I mean specifically the importance of the normal operator or normal matrix. You can also see the applications of this I mean spectral theorems for any other problems like to understand nature of the different form. Also one can use the concept of spectral theorems but I will try to give a very brief information about that in my next class which will be my last class in this course.