Advance Linear Algebra
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Lecture – 58
Normal Operator and Its Spectral Resolution

Welcome to the lecture series; we have seen if we consider a linear operator on a finite-dimensional inner product space, then if there exists an order basis B for this corresponding space V and the matrix representation of T is some upper triangular matrix. If the T is normal then the upper triangular matrix is basically a diagonal matrix. Now, the question is how to give a guarantee that there exists an orthonormal basis for the space of finite dimensional complex inner product space.

For, which operator will have a matrix representation as an upper triangular matrix? Today we are going to answer that question first. If I answer that question then I can immediately say that if the operator is normal matrix then that upper triangular will be diagonal matrix. Consequently, we can see the linear operator. The normal operator is diagnosable over that space.

I mean to say that the space will have a basis consisting of the eigenvector of the given operator T. So, let us quickly prove that result. That is what is the condition under which we can say that space V will have an orthonormal basis, such that the operator defined on that space will have an upper triangular matrix representation. So, it is like this.

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Theorem: Let V be a f.d complex i.p. space. Let T be a L.O m s.t There exist an orthonormal ordered basis for V such that the matrix representation of T w.r.to this basis is an upper Triangular matrix.

Pf: We shall prove this result by Induction on the dimension of the space V. Let dim of $V = n$. Assume the result is true for a f.d. complex i.p.s of dim $\leq (n-1)$.

Consider the adjoint opn T^* of T. Definitely there exist a non zero unit vector $\alpha \in V$ s.t

$$T^*\alpha = c\alpha \quad \text{for some } c \in \text{field of scalar m s.t}$$

Theorem:- Let V be a finite dimensional, complex inner product space T be a linear operator on it. Then there exists an orthonormal order basis for V such that the matrix representative T with respect to basis is an upper triangular matrix. So, the proof what we will do it here by induction on the order of the dimension of the space. Proof:- So, we shall prove this result by induction on the dimension of the space V.

Let dimension of V = n, assume the result is true for a finite-dimensional complex inner product space of dimension $\leq$ (n-1). So, consider the adjoint operator T^* of T. So, definitely, there exists a non-zero vector, I will say non-zero unit vector without loss of generality unit vector $\alpha \in V$ such that $T^*\alpha = c\alpha$ for some c $\in$ to the field of scalar on which V is defined.

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Consider W as a subspace, which is orthogonal complement of the subspace spanned by α. Note that the subspace spanned by α is T^* invariant.

$\Rightarrow W$ will be T-invariant

Consider S as a restriction of T on W

$\Rightarrow S$ is a L.O on W of dim $(n-1)$

$\therefore$ According to our hypothesis there exist an orthonormal ordered basis, B for W, such that the matrix representation of S w.r.to. it is an upper triangular matrix.

Now, consider $\beta' = \{\beta, \alpha\} = \{\alpha_1, \alpha_2 \cdots \alpha_{n-1}, \alpha\}$

where $\beta = \{\alpha_1, \alpha_2, \cdots \alpha_{n-1}\}$

Then we have $(T)_{\beta'} = \begin{bmatrix} \ddots & \\ 0 & \end{bmatrix}$

Consider W as a subspace of space, which is an orthogonal complement of the subspace spanned by α. Note that the subspace span by α is T^* invariant $\Longrightarrow$ W will be T invariant, according to our previous results. Consider S as a restriction of T on W $\Longrightarrow$ S is a linear operator on W of dimension (n–1).

Because the dimension of that subspace spanned by only a single element α is 1 and the dimension of this space V = n. So, the dimension of the orthogonal complement of that subspace will be (n–1), and the dimension of the W will be (n–1). So, according to our hypothesis, there exists an orthonormal order basis for W such that the matrix representation of S with respect to it is an upper triangular matrix.

Now, consider $B' = \{B, \alpha\} = \{\alpha_1, \alpha_2, ...\ \alpha_{n-1}, \alpha\}$ where $B = \{\alpha_1, \alpha_2, ...\ \alpha_{n-1}\}$. So then we

have $[T]_{B'} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ 0 & a_{22} & a_{23} & a_{2n} \\ 0 & 0 & a_{33} & a_{3n} \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$ because if you see the definition of the matrix

representations, $T\alpha = c\alpha$ other with the last row.

So, it will be the last row so and the first (n–1) rows is coming as a consequence from the matrix representation of your S. Since S is the restriction of T on W. So, therefore, I will have the matrix representation of T with respect to this new order basis B' will be like this. So, this, gives the proof that there is this an orthonormal basis under which the operator T will have an upper triangular matrix.

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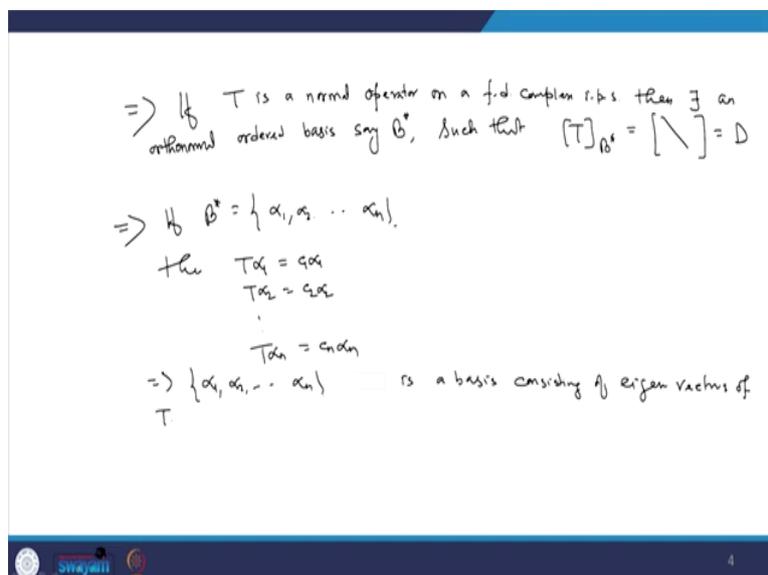
$\Rightarrow$ I can say if T is a normal operator on a finite-dimensional complex inner product space, then there exists an orthonormal order basis, say B*, such that the matrix representation of

$$[T]_{B^*} = \begin{bmatrix} a_{11} & 0 & 0 & 0 \\ 0 & a_{22} & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & a_{nn} \end{bmatrix} = D.$$ Because if the matrix representation of a normal

operator with respect some order basis is upper triangular.

Then that certainly that upper triangular is a diagonal matrix. Therefore, I can say that the normal operator T will have a basically diagonal matrix representation. So, this implies, if B* = $\{\alpha_1, \alpha_2, ... \alpha_n\}$ then $T\alpha_1 = c_1\alpha_1, T\alpha_2 = c_2\alpha_2, ... T\alpha_n = c_n\alpha_n \Rightarrow \{\alpha_1, \alpha_2, ... \alpha_n\}$ is the basis Of course orthonormal basis consist of consisting of eigenvectors of T.

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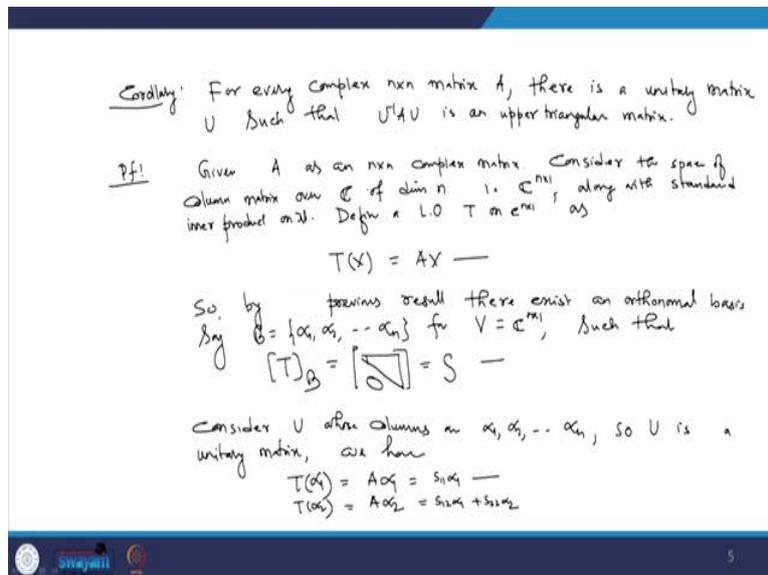
So now, let me come to some small corollary type. So, I can also come to the conclusions of the my last theorems that existence of order basis under which the operator can be written as an upper triangular matrix. Consequence of these results that for every complex nxn matrix A there is a unitary matrix U such that $U^{-1}AU$ is an upper triangular matrix.

So, the prove is exactly same what I did in the case of self-adjoint operator case. What I will do? I am quickly give the proof for this proof it is like this. Proof:- Given A as nxn complex matrix. So, consider the space of column matrix over complex plane $\mathbb{C}$ of dimension i.e. $\mathbb{C}^{nx1}$ along with standard inner product on it define a linear operator T on $\mathbb{C}^{nx1}$ as T(X) = AX.

I mean to say $[T]_B$ = A. So, by previous results there exist an orthonormal basis say B =

$\{\alpha_1, \alpha_2, \dots \alpha_n\}$ for this space V = $\mathbb{C}^{n\times 1}$ such that $[T]_B = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ 0 & a_{22} & a_{23} & a_{2n} \\ 0 & 0 & a_{33} & a_{3n} \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$ = S, it is

upper triangular matrix like this thing.

Then consider U which columns are $\{\alpha_1, \alpha_2, \dots \alpha_n\}$. So, U is a unitary matrix. We have here $T(\alpha_1) = A\alpha_1 = S_{11}\alpha_1$ because it is an upper triangular matrix. So, $T(\alpha_2) = A\alpha_2 = S_{12}\alpha_1 + S_{22}\alpha_2$. According to the definition of the matrix representations, I will certainly will have like these things.

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So, in this way we will see that your $A\alpha_n = \sum_{i=1}^{n} S_{in} \alpha_i \Rightarrow AU = US \Rightarrow U^{-1}AU = S$. So that is the proof. So, we see that one can also have unitary matrix such that $U^{-1}AU = S$ it is an upper triangular matrix.

This is also important results. So now, we have seen that a normal operator defined over finite dimensional complex inner product space is diagnosable. I mean the space V on which the normal operator is defined as in order basis which is orthonormal and consists of eigenvectors of the normal vector T. So, based on this results, let us write down one nice theorems which is basically conclusion of the whatever we have done.

So, let me summarize what we have done so, far regarding the normal operator. So, we have seen the normal operator defined over a finite dimensional inner product space. Of course, there
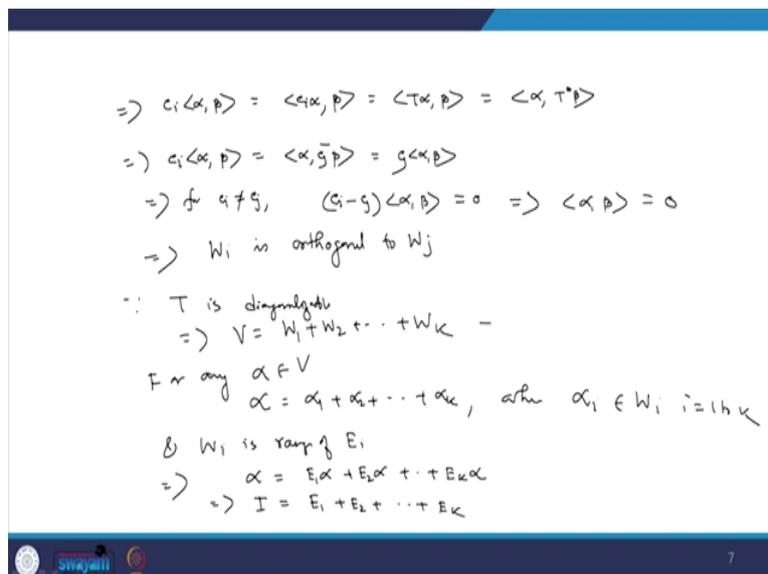
has to be complex then corresponding space will have order basis which is orthonormal. And I consist of eigenvector of the operator T. So, based on this results, let me talk about one nice theorem is called spectral theorem.

Spectral Theorems :- Let T be a normal operator on a finite dimensional complex inner product space V or a self-adjoint operator on a finite dimensional real inner product space and $c_1, c_2, \ldots c_k$ be the distinct eigenvalues of T. Let W_j be the characteristic space associated to eigenvalue c_j & E_j, the orthogonal projection of the space V on W_j. Then W_j is orthogonal to W_i for j $\neq$ i.

V is the direct sum of $W_1, W_2, \ldots W_k$ & the operator T = $c_1 E_1 + c_2 E_2 + \ldots + c_k E_k$ which is called the spectral resolution of the operator T on the space V. So, let me give the proof of this results, already we have done similar type of problems, long back in this course. So, it is basically repetitions so, quickly we will recall what we have done? And will also easy to understand this nice results.

So, we have to show that if j $\neq$ I then W_i is orthogonal to W_j. Let $\alpha \in W_i$ & $\beta \in W_j$. Let T$\alpha = c_i \alpha$, & T$\beta = c_i \beta$.

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$\Rightarrow c_i < \alpha, \beta > = < c_i \alpha, \beta > = < T\alpha, \beta > = < \alpha, T^*\beta > \Rightarrow c_i < \alpha, \beta > = < \alpha, \bar{c_j} \beta > = c_j < \alpha, \beta > \Rightarrow$ for $c_i \neq c_j$, we have $< c_i\text{-}c_j > < \alpha, \beta > = 0 \Rightarrow < \alpha, \beta > = 0 \Rightarrow W_i$ is orthogonal to W_j. Apart from this, we will also have since T is diagnozable, so, we already

know that $V = W_1 \oplus W_2 \oplus \ldots \oplus W_k$ and for any $\alpha \in V$ will have $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_k$ where $\alpha_i \in W_i$, i = 1 to k & W_i is range of $E_i \implies \alpha = E_1\alpha + E_2\alpha + \ldots + E_k\alpha$ because $E_i\alpha = \alpha_i$. So, based on that principles, I can say α equal like this.

So, this implies the identity operator, $I = E_1 + E_2 + \ldots + E_k$ & here this prediction operator E_i so, is a satisfied $E_iE_j = 0$ when $i \neq j$ & $E_i^2 = E_i$ for i = j.

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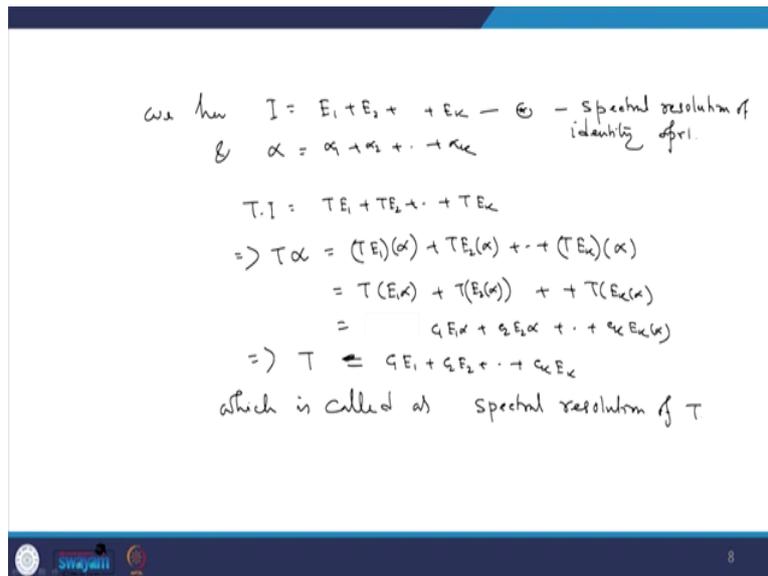
So, we have , $I = E_1 + E_2 + \ldots + E_k \rightarrow (*)$spectral resolution of identity operator & $\alpha = \alpha_1 + \alpha_2 + \ldots + \alpha_k$. So, if I multiply T to this equation (*), so I will be, $T.I = TE_1 + TE_2 + \ldots + TE_k$ $\implies T\alpha = (TE_1)\alpha + (TE_2)\alpha + \ldots + (TE_k)\alpha = T(E_1(\alpha)) + T(E_2(\alpha)) + \ldots + T(E_k(\alpha)) = c_1E_1\alpha + c_2E_2\alpha + \ldots + c_kE_k\alpha \implies T = c_1E_1 + c_2E_2 + \ldots + c_kE_k$ because $E_i\alpha \in W_i$.

which is called as spectral resolution of identity operator. See, Interestingly, we can immediately check that all the projection operator is basically polynomial how say it is like this.

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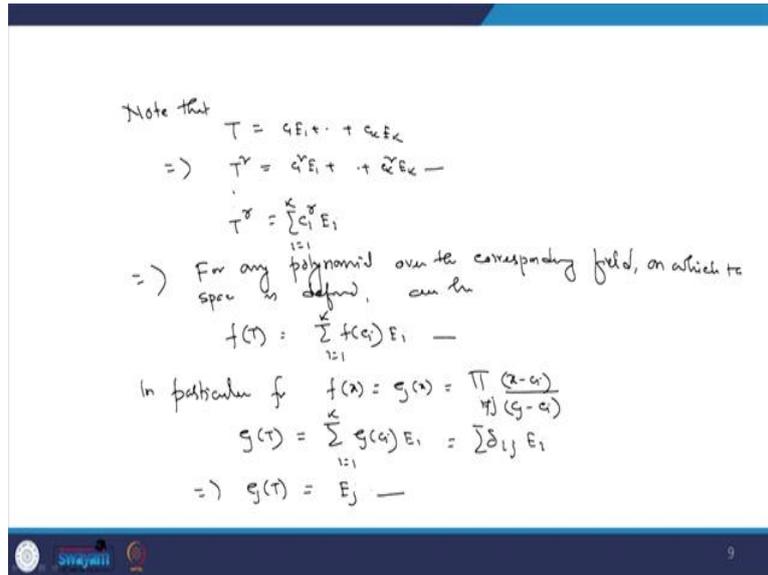
Note that we have, $T = c_1E_1 + c_2E_2 + ... + c_kE_k$, $T^2 = c^2{}_1E_1 + c^2{}_2E_2 + ... + c^2{}_kE_k$, ... $T^r = \sum_{i=1}^{k} c_i{}^r E_i$ $\Longrightarrow$ For any polynomial over the corresponding field on which the space is defined we have $f(T) = \sum_{i=1}^{k} f(c_i)E_i$

In particular when $f(x) = e_i(x) = \prod_{i \neq j} \frac{(x - c_i)}{(c_j - c_i)}$ Lagrange, interpolant polynomial,. So, in this case will have what see this polynomial having nature such that e_j at $c_j = 1$, e_j at any other value of c_i, y = 1 to k will have equal to 0. So, for choosing this type of typical polynomials, I will have $e_j(T) = \sum_{i=1}^{k} e_j(c_i)E_i = \sum_{i=1}^{k} \delta_{ij}E_i$ $\Longrightarrow e_j(T) = E_j$

So, because i = j then only I will have equal to 1, other is 0. So, the projection operator is basically polynomial in nature. So, this wonderful results will be used in different places of our analysis. In the case of any numerical calculations, numerical solution of the PDE has to use this concept of normal operators and this normal operator define over the finite dimensionl space which gives a set of orthonormal bases which is consist of eigenvectors.

This helps lot to even finding solutions, even it relies in singular value decomposance which I am going to do it in coming classes. So, you will see the beauty of this normal operator in our next class also. Thank you.