Advanced Linear Algebra
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Lecture – 57
Normal Operator – II

So, welcome to the lecture series on Advanced Linear Algebra. We have already seen for self-adjoint operator defined over a finite dimensional inner product space. The space has an orthonormal basis, consisting of the eigenvector of the operator defined on that space. Now, we are going to see the answer with a similar answer holds good when the self-adjoint operator is replaced by a normal operator.

But before moving to that, let us see what we have already seen from our last results. As a corollary of my previous result, I can say this: if A is an nxn Hermitian matrix that is a self-adjoint matrix, then there is a Unitary matrix, P, such that $P^{-1}AP$ is a diagonal matrix. That is, I can say that the original matrix A is self-adjoint is unitary equivalent to a diagonal matrix.

If A is a real symmetric matrix, then I will say that A is orthogonal equivalent to a diagram matrix. So, in that case, the P matrix will be an orthogonal matrix, and the $P^{-1}AP$ will be diagonal.

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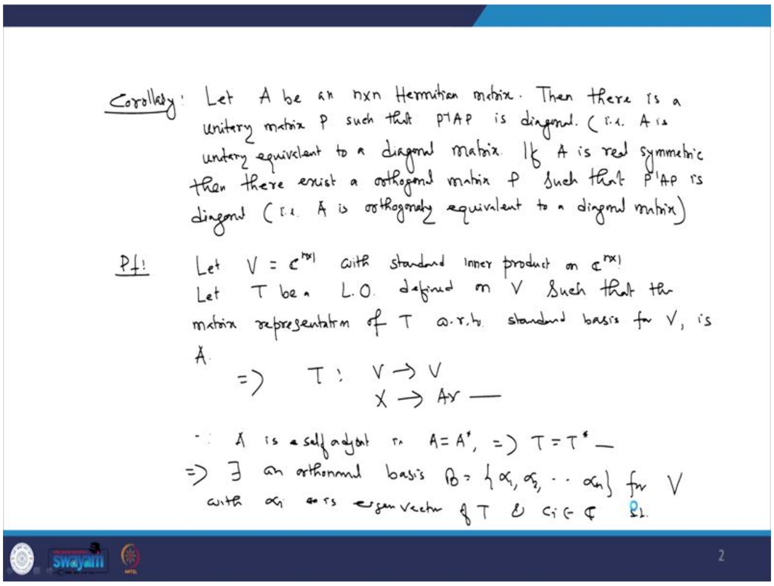

So, let us prove this results which is basically coming as a consequence of our previous theorems that self-adjoint operator define over a finite dimensional inner product space. That space has a basis orthonormal basis consisting of the eigenvector of the self-adjoint operator. So, let me consider V = $\mathbb{C}^{nx1}$ over the complex plane with a standard inner product on $\mathbb{C}^{nx1}$.

Let T be a linear operator defined on V such that the matrix representation of T, with respect to the standard basis for V, is A $\Longrightarrow$ T: V $\rightarrow$ V which is mapping X $\rightarrow$ AX. Now, since A is self-adjoint, that is A = A* $\Longrightarrow$ T = T*, i.e. T is also the self-adjoint operator $\Longrightarrow$ there exists an orthonormal basis, say, B = $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$ for the space V that is $\mathbb{C}^{nx1}$ with α_i, A is eigenvector of T.

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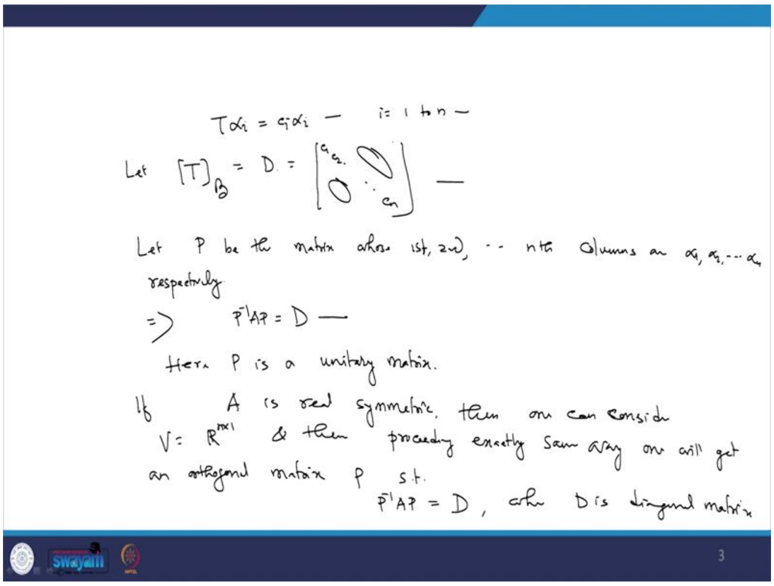

And c_i belongs to the corresponding field $\mathbb{C}$ such that Tc_i = $c_i\alpha_i$ for i = 1 to n. Let $[T]_B$ = D =

$$\begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & c_n \end{bmatrix}$$

is diagonal matrix. Let P be the matrix whose 1st, 2nd, … nth column are $\alpha_1, \alpha_2, \ldots\ \alpha_n$ respectively $\Longrightarrow$ $P^{-1}AP$ = D. Already we have proof that if an operator is diagnosable.

Here $\alpha_1, \alpha_2, \ldots\ \alpha_n$ are corresponding eigenvectors associated eigenvalues $c_1, c_2, \ldots\ c_n$. So then definitely, if I consider the P is a matrix in which columns are corresponding eigenvector, then definitely $P^{-1}AP$ = D. Here, P is a unitary matrix because here the $\alpha_1, \alpha_2, \ldots\ \alpha_n$ are defined over the complex plane C. So, P is here unitary matrix.

If A is real symmetric then one can consider V = R^{nx1} that is set of all nx1 column vector over the real line. Then, proceeding exactly the same way, one will get an orthogonal matrix P such that $P^{-1}AP$ = D it is a diagonal matrix.

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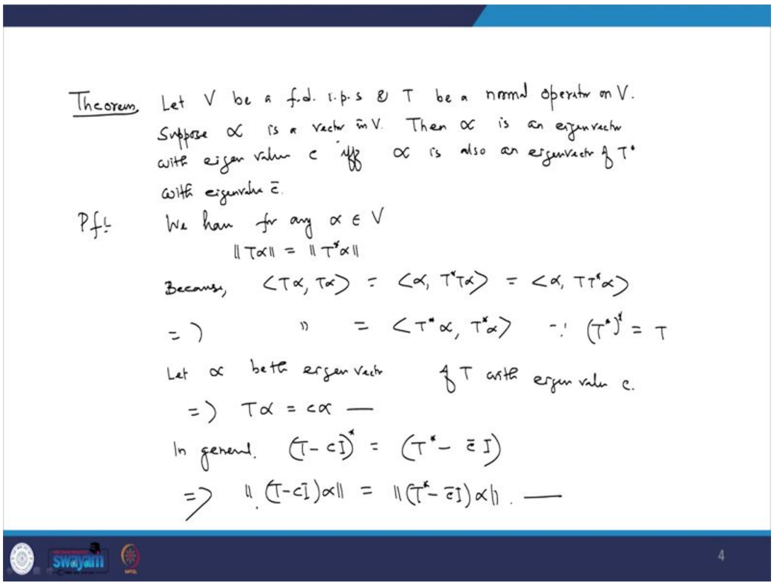

So, we have seen that if A matrix is Hermitian or self-adjoint then it is diagnosable. Now, let me consider the other situations. Other situation means now you want to drop the concept that operator is self adjoint. Now, if I consider instead of self-adjoint, simply normal operator then what is happening? So now, we want to replace the self-adjoint operator by normal operator. In this case, let us quickly check if you consider a normal operator defined over a finite-dimensional inner product space.

Then if α is vector in this space and if α is a eigenvector of T then with the eigenvalues, say c then definitely α will be also eigenvector of the corresponding adjoint operator with the eigenvalue $\bar{c}$. So, let us quickly prove this result. So, this will be act the key factors to prove our next results which is basically lead to our mission that if you consider normal operator on a finite dimensional inner product space.

This space will have an orthonormal basis consisting of the eigenvector of the normal operator. So, let us prove this result that V is a finite-dimensional inner product space and T is an normal operator on V. Suppose $\alpha$is a vector in V. Then $\alpha$is an eigenvector with eigenvalue c. If and only if α is an eigenvector of T^* with eigenvalue $\bar{c}$. We have for any $\alpha \in$ V, $\| T\alpha \| = \| T^*\alpha \|$.

Because now, $< T\alpha, T\alpha > = < \alpha, T^*T\alpha > = < \alpha, TT^*\alpha >$ since $T^*T = TT^*$, so, I can replace this T^*T by $TT^* \Longrightarrow < T\alpha, T\alpha > = < T^*\alpha, T^*\alpha >$ since $(T^*)^* = T$. So that is why $\| T\alpha\| = \| T^*\alpha\|$.

Now, if α is a eigenvector of T with eigenvalue say c $\Longrightarrow$ $T\alpha = c\alpha$. In general $(T - c\, I)^* = (T^* - \bar{c}\, I) \Longrightarrow$ I can say using the previous results, $\|(T\text{-}c\ I)\alpha \| = \| (T^* - \bar{c}\ I)\|$. For any $\alpha \in$ V, norm of this will be equal to this quantity.

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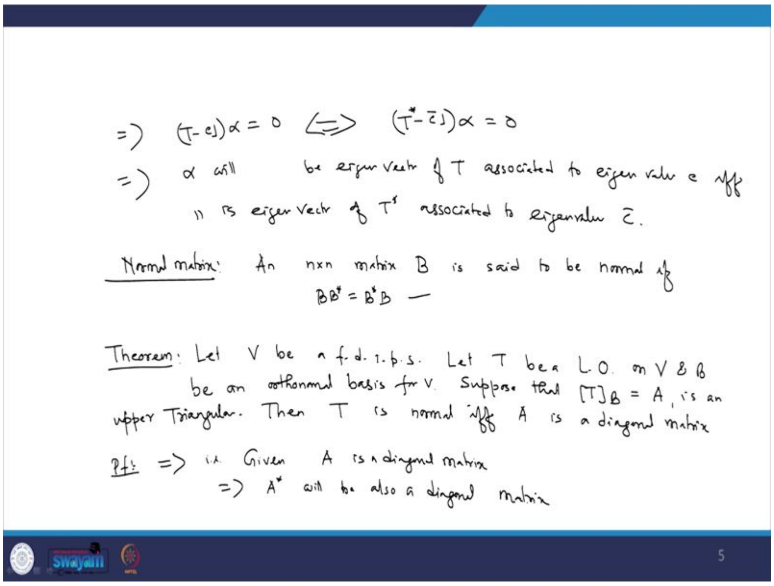

$\Longrightarrow$ (T-c I)α = 0 $\Leftrightarrow$ $(T^* - \bar{c}\ I)\alpha = 0 \Longrightarrow \alpha$ will be eigenvector of T associated to eigenvalue. See if and only if α is eigenvector of T^* associated to eigenvalue $\bar{c}$. So, we have seen that under what conditions, a non-zero vector of the space V will be eigenvector of the operator T is a normal operator.

So, let me define normal matrix also. I mean, as you have defined self-adjoint operator and self adjoint matrix here also, let me define normal matrix. Normal Matrix:- An nxn matrix B is said to be normal if $BB^* = B^*B$. So, a normal matrix is defined exactly same way the operator normal operator defined. Here I am saying that B will be normal, provided $BB^* = B^*B$.

Now, based on the definition of the normal matrix. Now, let me introduce one nice results. It is like this suppose V be a finite dimensional inner product space and T be a linear operator on V and B be orthonormal basis for V. Suppose $[T]_B$ = A is a upper triangular matrix. Then T is normal if and only if A is diagonal matrix.

So, the definition of a normal matrix or normal operator will be more interesting if we have understood this once. Otherwise simply saying that A or T commutes with this adjoint will not make that much of impressive, we cannot able to see, what is the inside of this operators? But if you consider this result, where I am showing that suppose say there is this a orthonormal basis on a finite dimensional inner product space V such that operator is defined over the space.

And the matrix representation of the operator is it upper triangular matrix then I am saying that T will be normal if and only if that upper triangular matrix has to be diagonal matrix. But this is the most important results if a normal operator is defined over a finite dimensional inner product space then space has an orthonormal basis consisting of eigenvector of the normal operator.

So, let us quickly prove this result. Let me give the proof the first if part that is given A is a diagonal matrix. This implies, I mean $A^*$is also so, A diagonal matrix.

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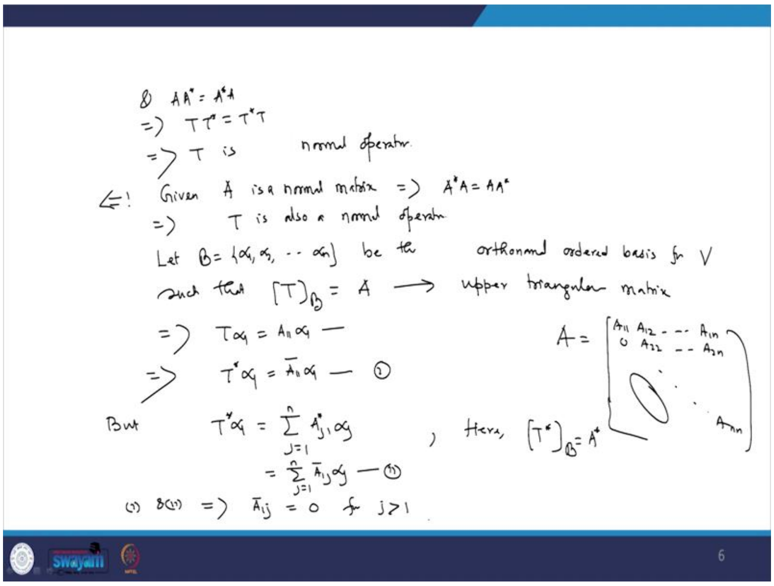

& $AA^* = A^*A \Longrightarrow TT^* = T^*T \Longrightarrow$ T is the normal operator. Now, let me consider the other way, given that given A is a normal matrix. So, this implies T is also a normal operator it is given to us. Let B = $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$ be orthonormal order basis for the space V such that $[T]_B$ = A =

$$\begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{1n} \\ 0 & A_{22} & A_{23} & A_{2n} \\ 0 & 0 & A_{33} & A_{3n} \\ 0 & 0 & 0 & A_{nn} \end{bmatrix} \rightarrow \text{(Upper triangular matrix)}$$

Because we know V is a finite dimensional space we can have a orthonormal basis. And the matrix representation of the T with respect to this basis will be suppose A which is also normal because T is normal $\Longrightarrow$ if you consider $T\alpha_1 = A_{11}\alpha_1 \Longrightarrow T^*\alpha_1 = \overline{A_{11}}\,\alpha_1 \rightarrow$ (i)

because the definition of the matrix representation of T with respect to order basis B says that the T of alpha 1 has to be basically $A_{11}\alpha_1 + A_{21}\alpha_2 + \ldots A_{n1}\alpha_n$. Because since this is upper triangular matrix $A_{21}, A_{31}, 0$.

1 according to our previous results. So, I am getting this once but according definition of the matrix representations of T^* with respect to B.

If I consider, $T^*\alpha_1 = \sum_{j=1}^{n} A^*{}_{j1}\,\alpha_j$ where $[T^*]_B = A^* \Longrightarrow T^*\alpha_1 = \sum_{j=1}^{n} \overline{A_{1j}}\,\alpha_j \rightarrow$ (ii) so, from (i) & (ii) $\Longrightarrow \overline{A_{1j}} = 0$ for $j > 1$. I mean $\overline{A_{12}} = 0$, $\overline{A_{13}} = 0$

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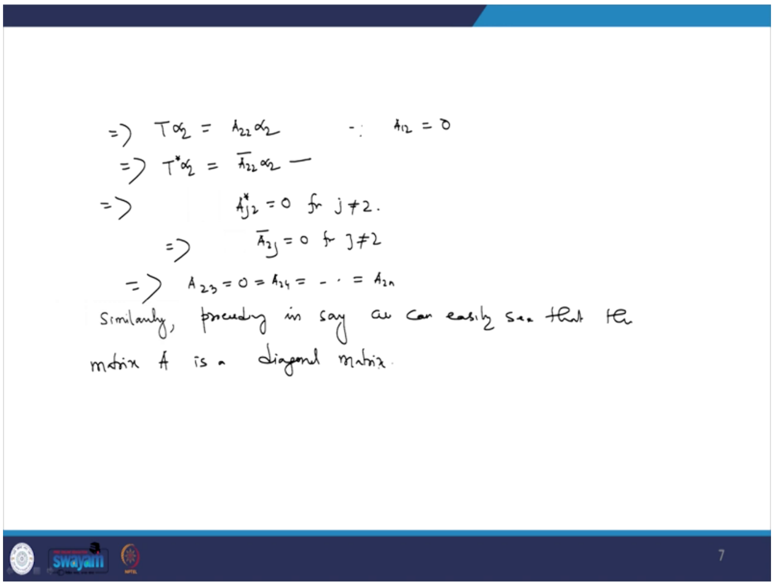

$\Longrightarrow T\alpha_2 = A_{21}\alpha_2 \Longrightarrow$ since $A_{12} = 0$ because $\overline{A_{12}} = 0 \Longrightarrow T^*\alpha_2 = \overline{A_{22}}\,\alpha_2 \Longrightarrow A^*{}_{j2} = 0$ for $j \neq 2 \Longrightarrow \overline{A_{2j}} = 0$ for $j \neq 2 \Longrightarrow A_{23} = 0 = A_{24} = 0 \ldots = A_{2n}$

Similarly, proceeding in the same way we can easily say that the matrix A is a diagonal matrix. So, what we see here is somehow, if I can have the matrix representation of a operator with respect to some orthonormal basis is a an upper triangular matrix. Then the operator T will be normal if and only if that upper triangular matrix is diagonal.

Now, this give a hints to show the existence of orthonormal basis for the space which are the eigenvectors of the normal operators. We are going to prove this result in our next class.