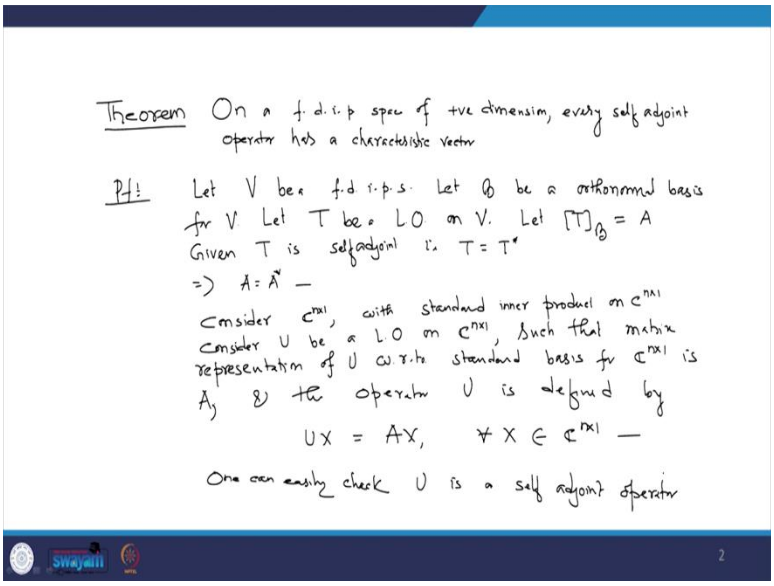
Advanced Linear Algebra
Prof. Premananda Bera
Department of Mathematics
Indian Institute of Technology – Roorkee

Lecture – 56
Normal Operator – I

Welcome to the lecture series on Advanced Linear Algebra. In my last class, I defined what a normal operator is. We have seen a special case of the normal operator, which is the self-adjoint operator. If it is an eigenvalue, then eigenvalues are real. But the existence of the eigenvectors is we have not answered. Now, let me answer these questions; suppose I consider a finite dimension on inner product space. And then my claim is that every self-adjoint operator has a characteristic vector.

(Refer Slide Time: 01:19)

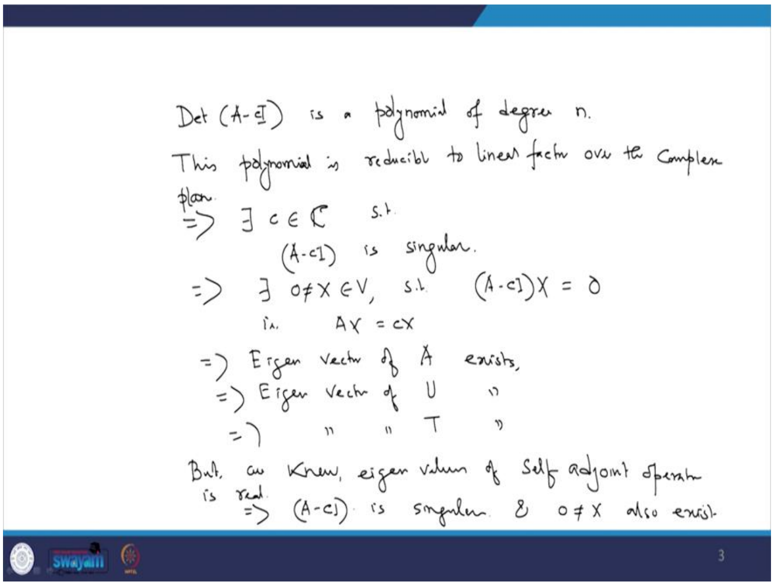


So, let me prove these results. Theorem: - Let V be a finite dimensional inner product space. Let B be an orthonormal basis this is for V. Let T be a linear operator on V and let $[T]_B$ = A, given T is self-adjoint i.e. T = T* ⟹ A = A*

Consider $\mathbb{C}^{nx1}$, with standard inner product on $\mathbb{C}^{nx1}$. Consider U be a operator a linear operator on $\mathbb{C}^{nx1}$ such that the matrix representations of U with respect to standard basis for $\mathbb{C}^{nx1}$ is A. So, I have considered that matrix representation of U with respect to standard basis on $\mathbb{C}^{nx1}$ is A.

& the operator U is defined by UX = AX ∀ X ∈ $\mathbb{C}^{nx1}$. One can easily check U is a self-adjoint operator because we have already done this in our previous lectures; the operator defined this as a self-adjoint. So that we can follow, or you can also take it as homework to check this new operator U on $\mathbb{C}^{nx1}$ is a self-adjoint operator.

(Refer Slide Time: 05:28)

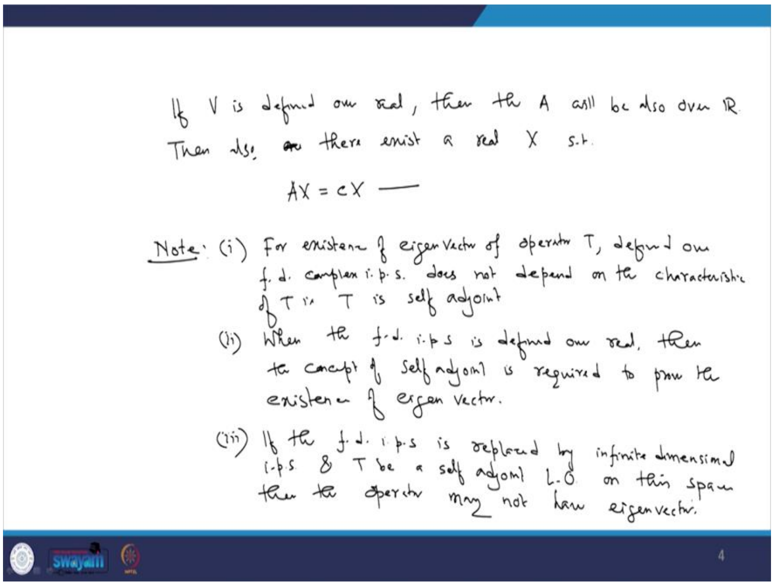


Now, Det(A-cI) is a polynomial of degree n. This polynomial is reducible to linear factors over the complex plane ⟹ ∃ c ∈ ℂ s.t. (A–cI) is singular ⟹ ∃ 0 ≠ X ∈ V, s.t. (A–cI)X = 0 i.e. AX = cX.

⟹ Eigenvector of the matrix A exists ⟹ Eigenvector of U exist ⟹ Eigenvector of T exist. But we know the eigenvalues of the self-adjoint operator are real ⟹ I mean, all the eigenvalues of the self-adjoint operator are real ⟹ (A–cI) is singular & 0 ≠ X also exist. Here it is important to say here.

If it is self-adjoint and all the eigenvalues are real, this implies the characteristic polynomial will have all the coefficients over real.

(Refer Slide Time: 08:34)

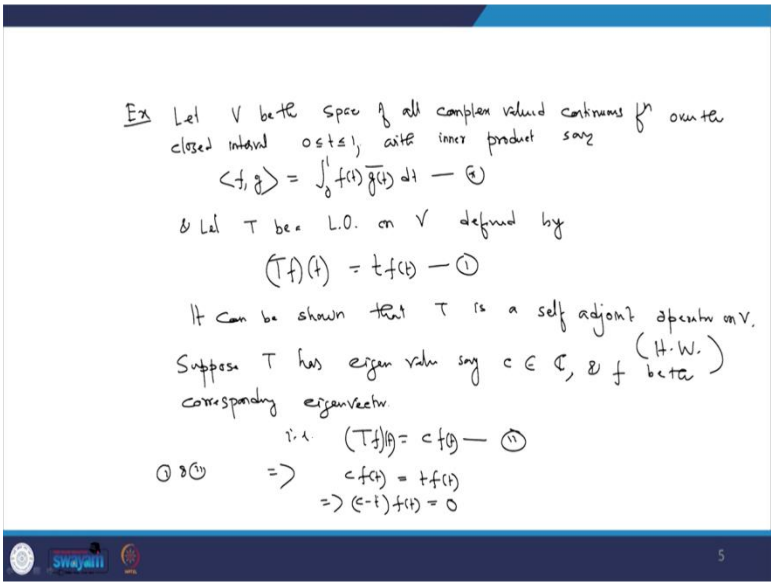


So, in this context, I can say if V is defined over real, then the matrix A will also be over ℝ. So then also there exists a real X, I mean, entries of the vector X will be over the real number. $\exists X \in \mathbb{R}$ such that AX = cX. So, in both cases, the existence of an eigenvector is guaranteed. See, let me point out three more important observations in this proof.

Note:- (i) For existence of the eigenvector of operator T defined over finite dimensional complex inner product space does not depend on the characteristic of T i.e. T is self-adjoint, I mean to say to prove the existence of an eigenvector for the operator T defined over finite dimensional complex inner product space, we do not need the concept T to be self-adjoint, however, (ii) when the finite-dimensional inner product space is defined over real.

then, the concept of self-adjoint is required to prove the existence of the eigenvector. (iii) we have shown the result only for finite-dimensional product space. If the finite-dimensional inner product space is replaced by infinite-dimensional inner product space & T be a self-adjoint linear operator on this space, then the operator may not have an eigenvector.

(Refer Slide Time: 12:51)

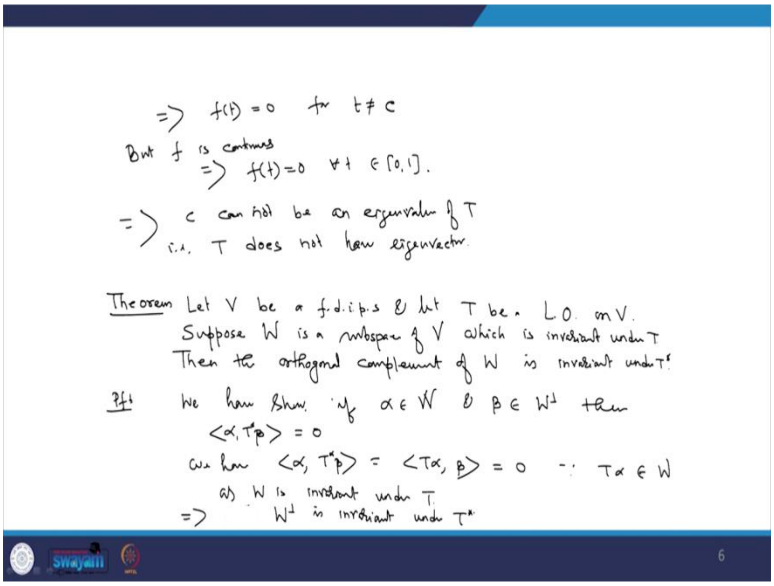


Example:- Let V be the space of all complex-valued continuous functions over the closed interval, say $0 \leq t \leq 1$ with $< f, g > = \int_0^1 f(t)\overline{g(t)}\, dt \rightarrow (*)$ this is my inner product & Let T be a linear operator on V defined by (Tf)(f) = t f(t) $\rightarrow$ (i) this is my operator.

Again, I will give you this as a homework assignment so that it can be shown that T is a self-adjoint operator on the space V, so that you can prove it as a considered homework. So, now, I will show that this operator does not have an eigenvector even though it is defined over an inner product space and is also self-adjoint, but it does not have any eigenvector. So, suppose T has an eigenvalue, saying c ∈ ℂ.

And f be the corresponding eigenvector i.e (Tf)(t) = c$f(t)$ $\rightarrow$ (ii) this is my linear operator. So from (i) & (ii) we have, cf(t) = t f(t) $\Longrightarrow$ (c-t)f(t) = 0.

(Refer Slide Time: 16:27)

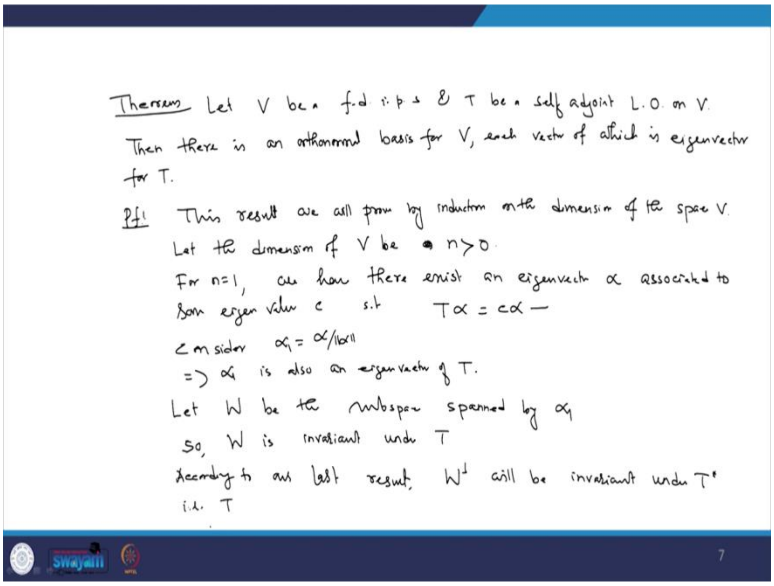


$\Longrightarrow$ f(t) = 0 for t $\neq$ c but f is continuous $\Longrightarrow$ f(t) = 0, for all t $\in$ [0, 1] $\Longrightarrow$ c cannot be and eigenvalue of T i.e. T does not have eigenvector. So, we have seen that if the self-adjoint operator is defined over inner product space, then the operator may not have eigenvectors. So now, let me prove one more, nice result; it is like this.

Let V be a finite-dimensional inner product space & let T be a linear operator on V. Suppose W is a subspace of V, which is invariant under T. Then the orthogonal complement of W is invariant under T^*. The proof of this result is very simple. Proof:- we have to show if $\alpha \in$ W & $\beta \in W^{\perp}$, then $< \alpha, T^*\beta > = 0$ where T^* is basically the adjoint of T.

We have, $< \alpha, T^*\beta > = 0 = < T\alpha, \beta > = 0$ since $T\alpha \in$ W, as W is invariant under T. So, since W is invariant under T for any element, $\alpha \in$ W, $T\alpha \in$ W. So, therefore $< T\alpha, \beta > \in W^{\perp}$ this is implies $< T\alpha, \beta > = 0$. So, this implies what $W^{\perp}$ is invariant under T^*?

(Refer Slide Time: 21:00)



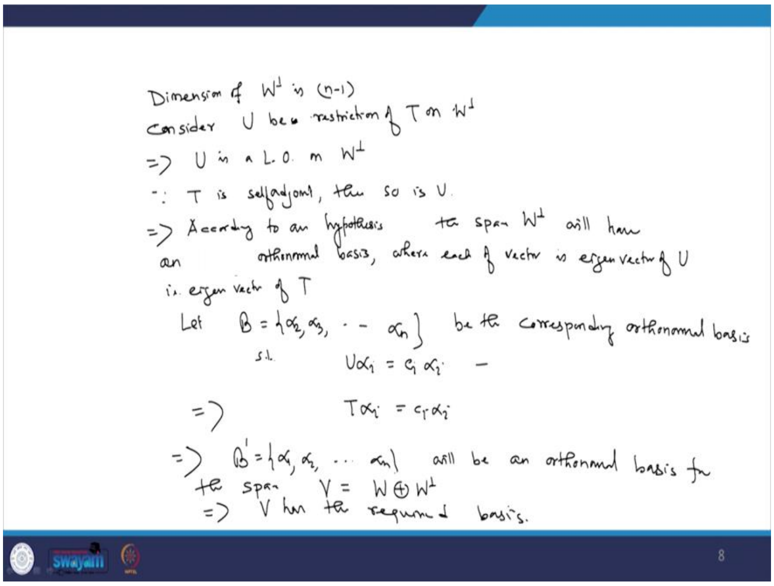
Let me consider another nice result, so now, we are going to provide the same answer to our question, which I raised in my last lecture. That is, if you consider a finite-dimensional inner product space and if a T is a linear operator defined on it. Then, under what conditions the space will have an orthonormal basis in which each vector is basically the eigenvector of the corresponding operator?

To answer that question only we introduce the concept of normal operator. Again, we came to this special case, which is a self-adjoint operator, which is also the normal operator. Now, I will show that if I consider it a self-adjoint operator on a finite-dimensional inner product space. Then, the space has an orthonormal basis in which each vector is an eigenvector of the self-adjoint operator.

So, which is given this theorem, see it like this: V is a finite dimensional inner product space. If T is a linear operator on V, then there is the orthonormal basis for V, each vector of which is the eigenvector for T. Proof: - This result will be proved by induction on the dimension of the space V. Let the dimension of V be n > 0. For n=1, we have according to our previous results, there exists and eigenvector α associated to some eigenvalue s.t. T$\alpha = c\alpha$

Consider $\alpha_1 = \frac{\alpha}{||\alpha||}$ $\Longrightarrow$ α_1 is also an eigenvector of T. Let W be the subspace spanned by α_1. So, W is invariant under T. So, according to our last result, $W^{\perp}$ will be invariant under T^* i.e. T = T^*.

(Refer Slide Time: 26:44)



So, the dimension of $W^{\perp}$ is (n–1) and considers U to be a restriction operator restriction of T on W⊥. ⟹ U is a linear operator on W⊥ since T is self-adjoint, then so is U then U also will be self-adjoint on $W^{\perp}$ ⟹ According to our hypothesis, what is the hypothesis you have considered? I have assumed that suppose the result is true for any space inner product space of dimension < n.

So now, we have to prove that the result is also true for dimension n. So, since the dimension of $W^{\perp}$ is (n–1). I can assume that the result is true for the operator U which is defined over an inner product space of dimension (n–1) and definitely will have an orthonormal basis consisting of the eigenvector of the operator U. Now, according to the hypothesis, the space $W^{\perp}$ will have an orthonormal basis where each of vectors is eigenvector of U.

i.e., an eigenvector of T. So, let B = $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ which has (n–1) elements be the corresponding orthonormal basis such that Uα_i = $c_i\alpha_i$ ⟹ Tα_i = $c_i\alpha_i$ ⟹ B′ = $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ will be an orthonormal basis for the space V = W ⊕ $W^{\perp}$.

This is a direct sum of W, and the orthogonal complement of W ⟹ V has the required basis. So, we have shown here that the inner product space V has an orthonormal basis consisting of the eigenvector of the operator T. So, we have shown for this self-adjoint operator. However, we have not shown what will happen if we replace the self-adjoint operator with a normal operator. So, these questions will be answered in our next lecture.