Advanced Linear Algebra
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Lecture – 55
Application of Unitary Operator and Initiation of Normal Operator

Welcome to lecture series of Advanced Linear Algebra. In my last class we have shown for every n cross n square invertible matrix there exist unique lower triangular matrix with integer on the main diagonals are positive and also a unitary of matrix, such that the unitary matrix equal to lower triangular matrix multiplied to the; you given invertible matrix. So now, let me consider an example so that how to obtain the lower trigonometric that will be clear.

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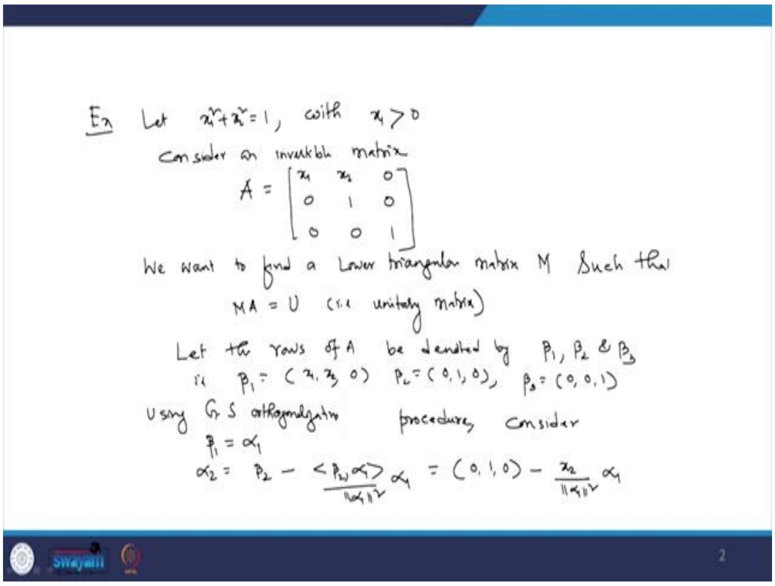

Ex: - Let $x_1{}^2 + x_2{}^2 = 1$, $x_1 > 0$. So, consider an invertible matrix, say A = $\begin{bmatrix} x_1 & x_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. So that I am confirm the given matrix is invertible. We want to find a lower triangular matrix say M such that MA = U (i.e. unitary matrix). How to find the lower triangular matrix?

So, I have to follow the proof how the matrix M has been constructed. So, let me go to the Gram Schmidt orthogonal procedures. Let the rows of A be β_1, β_2, β_3 i.e. $\beta_1 = (x_1, x_2, 0\,)$, $\beta_2 = (0, 1, 0)$, $\beta_3 = (0, 0, 1)$ using Gram-Schmidt orthogonalization procedure. Consider $\alpha_1 = \beta_1$ & $\alpha_2 = \beta_2 - \frac{<\beta_2, \alpha_1>}{||\alpha_1||^2}\,\alpha_1 = (0, 1, 0) - \frac{x_1}{||\alpha_1||^2}\,\alpha_1$

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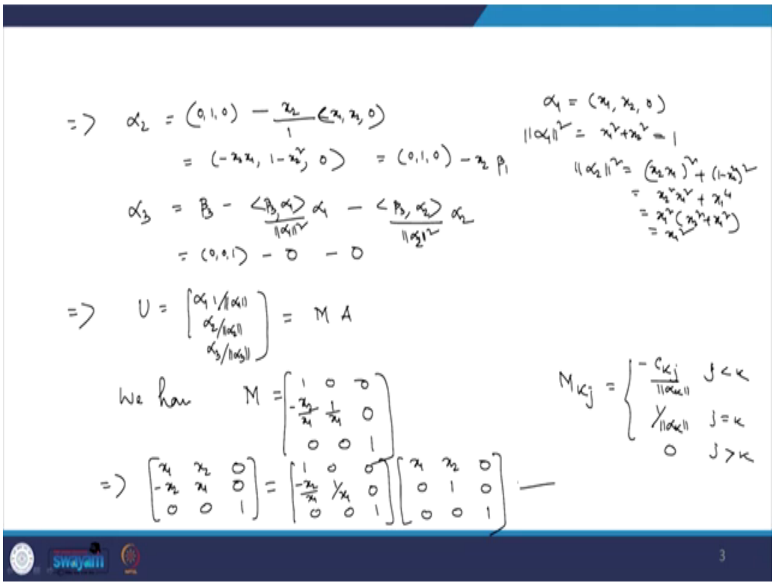

$\alpha_2 = (0,1,0) - \frac{x_2}{1}(x_1, x_2, 0)$ where $\alpha_1 = (x_1, x_2, 0)$ & $||\alpha_1||^2 = x_1{}^2 + x_2{}^2 = 1$ & $\alpha_3 = \beta_3 - \frac{<\beta_3, \alpha_1>}{||\alpha_1||^2}\alpha_1 - \frac{<\beta_3, \alpha_2>}{||\alpha_2||^2}\alpha_2 = (0,0,1) - 0 - 0$ where $||\alpha_2||^2 = (x_2 x_1)^2 + (1 - x_2{}^2)^2 = x_2{}^2 x_1{}^2 + x_1{}^4 = x_1{}^2(x_2{}^2 + x_1{}^2) = x_1{}^2 \implies U = \begin{bmatrix} \frac{\alpha_1}{||\alpha_1||} \\ \frac{\alpha_2}{||\alpha_2||} \\ \frac{\alpha_3}{||\alpha_3||} \end{bmatrix} = MA$, where U is a unitary matrix. So, what is our M? Here we have M will be equal to according to the definitions, what we have already learned in our last lectures, so, the coefficient that is if we call we know $M_{k_j} = \begin{cases} \frac{-1}{||\alpha_k||} c_{k_j} \; for\; j < k \\ \frac{1}{||\alpha_k||} \quad for\; j = k \\ 0 \; for\; j > k \end{cases}$

We have M = $\begin{bmatrix} 1 & 0 & 0 \\ \frac{-x_2}{x_1} & \frac{1}{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies \begin{bmatrix} x_1 & x_2 & 0 \\ -x_2 & x_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{-x_2}{x_1} & \frac{1}{x_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(()) (09:42) You can cross check whether this calculation is correct or not and, but the procedure is like this. So, in this way you can find out in what triangular matrix for any given nxn invertible complex matrix. So, now we will turn to this some another interesting fact by arranging a question.

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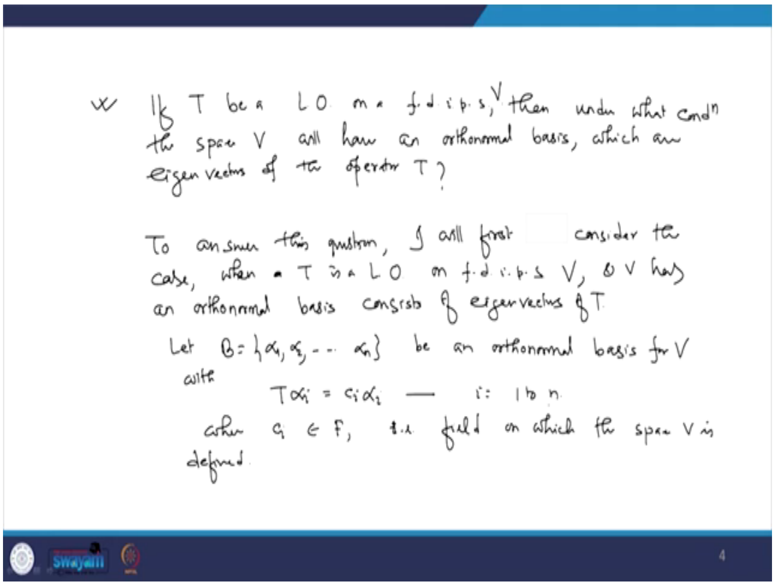

The question is like this if T be a linear operator on a finite dimensional inner product space. Then under what condition in a product space a V will have an orthonormal basis which are eigenvector of the operator T? So, let me raise this questions that if you picked off any linear operator on a finite dimensional inner product space say V.

Then, under what condition, the space V will have an orthonormal basis which are basically eigenvectors of the given operator now to answer this question. I will first consider suppose a linear operator T is defined on in finite dimensional inner product space and the space has an orthonormal basis which are basically eigenvectors of the operator T. Then, what sort of characteristics of the operator 1 can find it?

So, let me cross check that one first. So that will give a hint to answer these questions. So, to answer these questions, I will first consider the case when T is a linear operator on finite dimensional inner product space V and V has an orthonormal basis consists of eigenvectors of T. So, I have taken the reverse way. Suppose the operator T is defined on an inner product space and the space has an orthogonal basis consisting of the eigenvectors of the T.

Then I want to know, what is the typical characteristic one may see on this operator? Maybe that will give a hint to answer the above questions. So, I have taken an operator T which defined over a finite dimensional space inner product space V. So, let B = $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ be an orthonormal basis for V with T(α_i) = $c_i\alpha_i$, where c_i belongs to the corresponding field.

If that is field on which the space V is defined. So, I mean to say here c_i is basically eigenvalue of the operator T and i = 1 to n.

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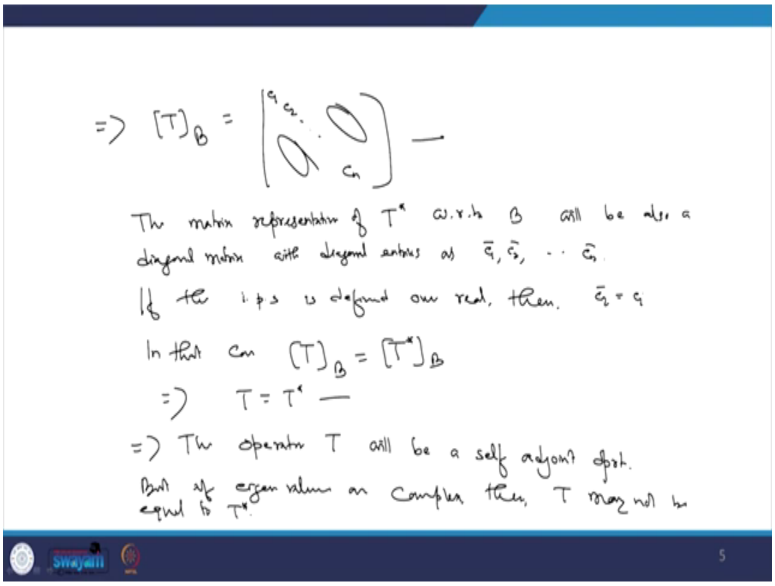

$[T]_B = \begin{bmatrix} c_1 & 0 & 0 & 0 \\ 0 & c_2 & 0 & 0 \\ 0 & 0 & c_3 & 0 \\ 0 & 0 & 0 & c_n \end{bmatrix}$. Now, the question is what about the adjoint of T with respect to same order. Since, B is finite dimensional so, operator T will certainly have adjoint. The matrix representation of T^* with respect to B will be also a diagonal matrix with diagonal entries as $\overline{c_1}, \overline{c_2}, \ldots \overline{c_n}$

If the inner product space is defined over real, then $\overline{c_i} = c_i$. So, in that case that case $[T]_B = [T^*]_B \implies T = T^* \implies$ The operator will be a self-adjoint operator. But if eigenvalues are complex T may not be equal to T^*.

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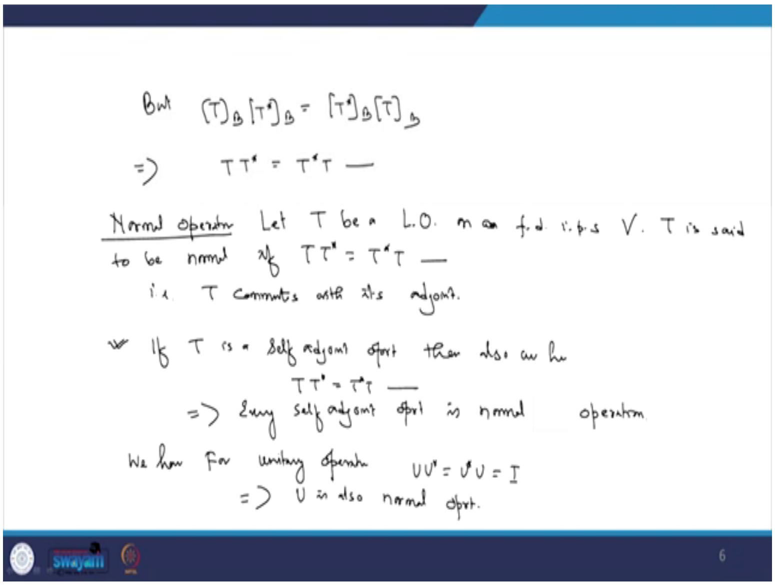

But $[T]_B\,[\mathrm{T}^*]_B = [\mathrm{T}^*]_B[T]_B \Longrightarrow \mathrm{TT}^* = \mathrm{T}^*\mathrm{T}$. So, now we have seen when a linear operator defines on a finite dimensional inner product space. And the space has a orthonormal basis consisting of eigenvector of the operator.

Then there are two situations when the eigenvalues are basically real. Then in that case I am getting that the operator is self-adjoint operator where $\mathrm{T}^* = \mathrm{T}$. But if it is eigenvalues are not real, in that case, I am not saying that T^* will be equal to T but certainly I will have $\mathrm{TT}^* = \mathrm{T}^*\mathrm{T}$. Now, this will give a hint, and this is a very fantastic hint that it is also sufficient condition.

If the operator T satisfy this condition, there certainly will have a orthonormal basis of the space consisting of the eigenvector of the operator T. I will give the proof later we are going to prove that that part later. But before going to the proof of that part, let me introduce a nice operator called a normal operator. Normal Operator: - Let T be a linear operator on a finite dimensional inner product space V, T is said to be normal if $\mathrm{TT}^* = \mathrm{T}^*\mathrm{T}$.

One may have a question in mind, while defining normal operator. In this way, I have taken that the operator defined over a finite dimensional space, why it is? Why have we restricted to the finite dimensional inner product space? The reason is, if you consider finite dimensional inner product space existence of the adjoint is guaranteed. So, based on that only I have defined it here that a linear operator T on a finite dimensional inner product space will be normal if $\mathrm{TT}^* = \mathrm{T}^*\mathrm{T}$.

And the adjoint commutes each other i.e T commutes with it is adjoint. See as a special case if T is a self-adjoint then also we have $TT^* = T^*T$, the self-adjoint means $T^* = T$. So, $TT^* = T^2$, so, this property holds good $\Longrightarrow$ Every self-adjoint operator is normal operator. What about unitary operator? We have for unitary operator, $UU^* = U^*U = I$.

$\Longrightarrow$ Here also U commutes with adjoint. U is also normal operator, so, every unitary operator is also normal operator. Every self-adjoint operator is also normal operator. Before going to answer the question which I raised the very beginning, I will first try to answer the questions related to the special case. That is self adjoint operator.

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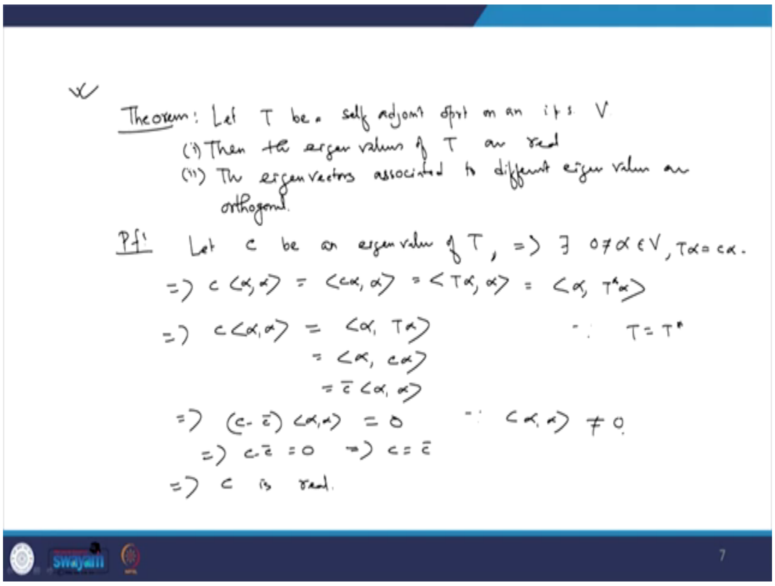

So, let me say or the characteristic of the self-adjoint operator. This is small theorems it is like this. Theorem: - Let T be a self-adjoint operator on an inner product space say V. Then, (i) The eigenvalues of T are real. (ii) eigenvectors associated to different eigenvalues are orthogonal. Proof: - Let c be an eigenvalue of T $\Longrightarrow \exists\, 0 \neq \alpha \in V$, $T\alpha = c\alpha$

$\Longrightarrow c<\alpha,\alpha> = <c\alpha,\alpha> = <T\alpha,\alpha> = <\alpha,T^*\alpha> = <\alpha,T\alpha>$ where $T^* = T$ $\Longrightarrow$ $c<\alpha,\alpha> = <\alpha,T\alpha> = <\alpha,c\alpha> = \bar{c}<\alpha,\alpha>$ $\Longrightarrow (c-\bar{c})<\alpha,\alpha> = 0$ since $<\alpha,\alpha> \neq 0$ $\Longrightarrow (c-\bar{c}) = 0$ $\Longrightarrow c = \bar{c}$ $\Longrightarrow$ c is real.

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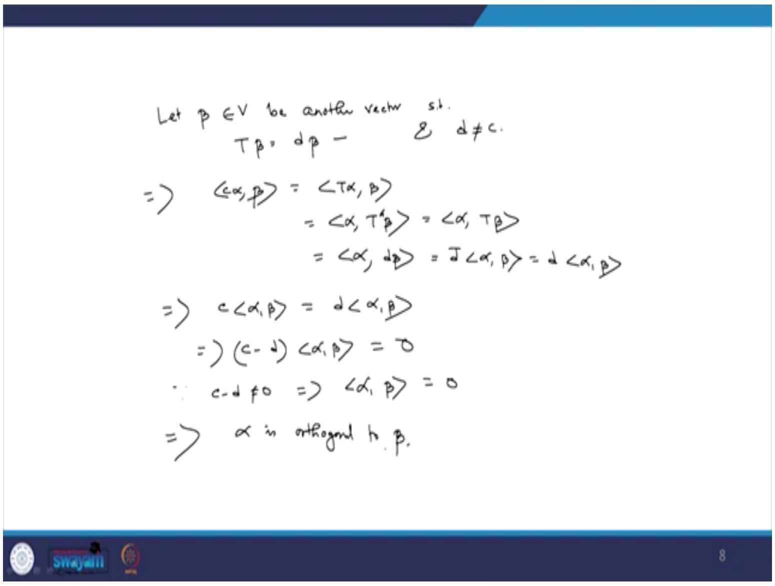

Let $\beta \in$ V be another vector such that Tβ = dβ & d $\neq$ c $\Longrightarrow < c\alpha, \beta > = <$ T$\alpha, \beta > = < \alpha,$ T$^*\beta > = < \alpha, T\beta > \Longrightarrow < \alpha, d\beta > = \bar{d} < \alpha, \beta > = d < \alpha, \beta >$ where $\bar{d} = d$ because all eigenvalue is real $\Longrightarrow$ c$< \alpha, \beta > = d < \alpha, \beta >$ (c– d) $< \alpha, \beta > = 0$ since (c-d) $\neq 0$ $\Longrightarrow$ $< \alpha, \beta > = 0$

$\Longrightarrow$ α is orthogonal to β. So, for different eigenvector associated to different eigenvalues are orthogonal to each other. Will continue more about it in our next class.