**Advanced Linear Algebra**
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**Lecture – 54**
**Unitary Operators - II**

Welcome to the lecture series on Advanced Linear Algebra. We have already seen in our last lecture what is Unitary Operator? What is a Unitary Matrix? We have seen over a finite-dimensional inner product space. The unitary operators form a group with respect to multiplication compositions. So, based on those principles if I consider, let U(n) denote set of all nxn unitary matrices. Then, U(n) is also a group with respect to multiplication composition.

$\Longrightarrow A, B \in$ U(n) & AB $\in$ U(n) & $A^{-1} \in$ U(n) and since each unitary operator is also an isomorphism $\Longrightarrow A^{-1} = A^{*} \in$ U(n). So, all these properties of the group also hold good in U(n). Based on this concept, only we shall proceed, and it will utilize this concept to prove some interesting results.

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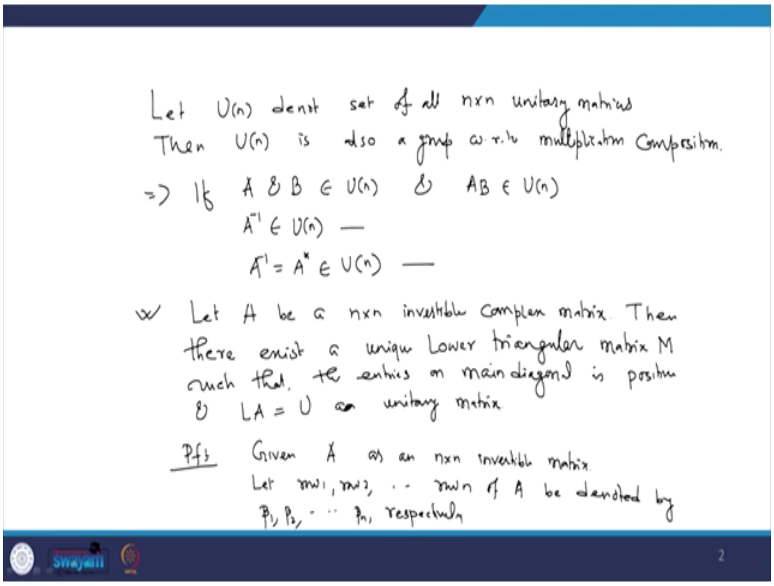


So, let me consider a nice problem. Let A be a nxn invertible complex matrix. Then my claim there exists a unique lower triangular matrix say M such that the entries or main diagonal is positive and LA = U a unitary matrix. So, what I want to say, if you consider any nxn invertible complex matrix. Then there exist a unique lower triangular matrix with the characteristic that each entry on the main diagonals are positive number.

And LA = U, a unitary matrix. So, let us give the proof of these results so proof of this result is based on basically Gram-Schmidt orthogonalization procedures. If you recall how one has to translate one has to get from a set of linear independent vectors to a set of linear independent orthogonal vectors by using the Gram-Schmidt orthogonalization procedure, then you will see the proof is very simple.

At least for the existence of the lower triangle and the uniqueness has to be also approved separately. So, let us see how to prove this result. Given A be a nxn invertible matrix. Let row 1, row 2, … , row n of A be denoted by $\beta_1$, $\beta_2$, … $\beta_n$ respectively.

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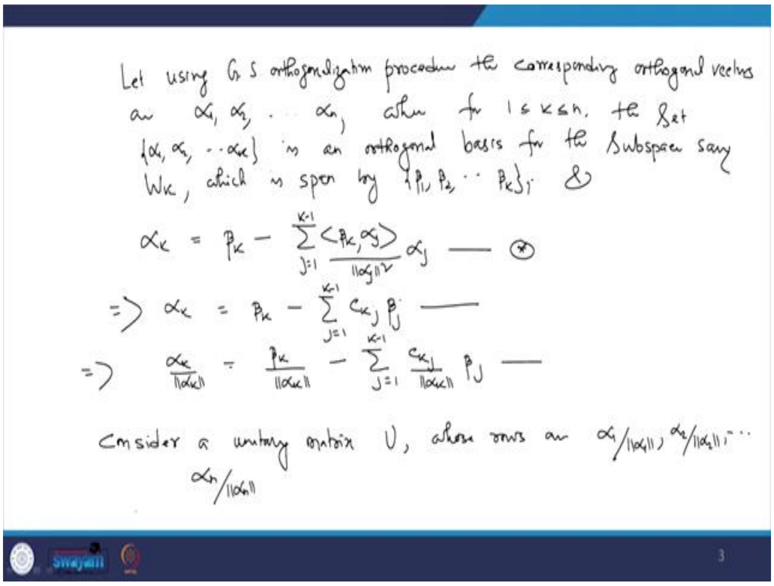


Let's use Gram-Schmidt orthogonalization procedures. The corresponding orthogonal vectors are $\alpha_1, \alpha_2, \ldots \alpha_n$ where for $1 \leq k \leq$n. The set $\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ is an orthogonal basis for the subspace say $W_k$ which is spanned by $\beta_1$, $\beta_2$, … $\beta_k$ and the kth orthogonal vector $\alpha_k = \beta_k$ $\sum_{i=1}^{k-1} \frac{<\beta_k, \alpha_j>}{||\alpha_j||^2} \alpha_j$

So, we know this Gram-Schmidt orthogonalization procedure. That procedure says that the k-th orthogonal vector will be of this form $\Longrightarrow \alpha_k = \beta_k - \sum_{j=1}^{k-1} c_{k_j} \beta_j$

What is the motivation behind considering this? My aim is that I want to write down the given matrix as a product of a lower triangular matrix and unitary matrix. So, based on that only I am rewriting this structure.

$\Longrightarrow \frac{\alpha_k}{||\alpha_k||} = \frac{\beta_k}{||\alpha_k||} - \sum_{j=1}^{k-1} \frac{c_{k_j}}{||\alpha_k||}\beta_j$, so, now consider a unitary matrix U whose rows are $\frac{\alpha_1}{||\alpha_1||}, \frac{\alpha_2}{||\alpha_2||}, \ldots \frac{\alpha_n}{||\alpha_n||}$ so that each of them will be orthonormal by which of this vector will be 1.

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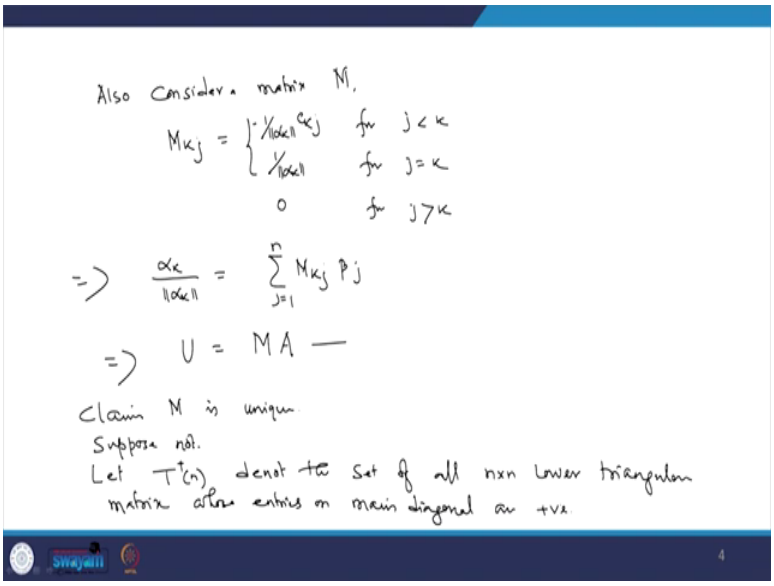


And let me consider also consider a matrix say M defined by where $M_{k_j} = \begin{cases} \frac{-1}{||\alpha_k||} c_{k_j} \; for \; j < k \\ for \; j = k \\ 0 \; for \; j > k \end{cases}$

So, the way I have considered the kth row of the matrix M. So, certainly $M_{k_j}$ will be gives me a lower triangular matrix and which main diagonal entries are if I consider main diagonal, entry is $M_{k_k}$.

These are basically $\frac{1}{||\alpha_k||} \Longrightarrow \frac{\alpha_k}{||\alpha_k||} = \sum_{j=1}^{n} M_{k_j}\beta_j$ $\Longrightarrow$ U = MA, So, this means that there exists a lower triangular matrix M such that U = MA.

Now, this M is also invertible will prove it later on, so that we can say that A = $\text{MU}^{-1}$. But anyhow, so, before going to that part, so, first, I have to show that this M is unique. Claim M each unique, suppose not. Let $\text{T}^{+}(n)$ denote the set of all nxn lower triangular matrices in whose entries on the main diagonal are positive. This means that each of the lower triangular matrices is invertible.

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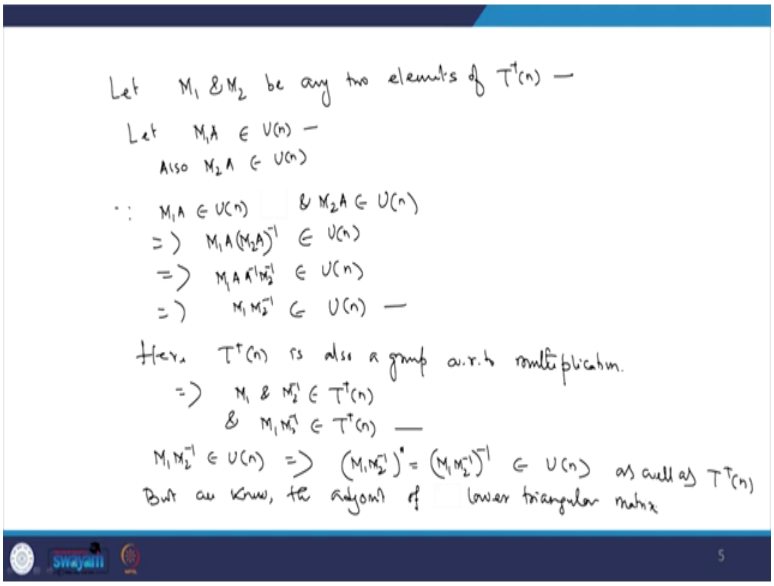


Let $M_1$ & $M_2$ be any two elements of $\mathrm{T}^+(n)$, I mean set of all the lower triangular matrix with the properties that main negative entries are positive. Let $M_1$A ∈ U(n), I mean to say this set of all nxn unitary matrix. Also $M_2$A ∈ U(n), I am saying there are two lower triangular say both $M_1$A & $M_2$A are unitary matrix claim.

$M_1$= $M_2$. Since $M_1$A ∈ U(n) & $M_2$A ∈ U(n) ⟹ $M_1\mathrm{A}(M_2\mathrm{A})^{-1}$ ∈ U(n) ⟹ $M_1\mathrm{A}A^{-1}{M_2}^{-1}$ ∈ U(n) ⟹ $M_1{M_2}^{-1}$ ∈ U(n), I have not going to prove that one nice results that $\mathrm{T}^+(n)$ that is set of all the lower triangular matrix with the diagonal entries are positive. They form a group with respect to multiplications that part I am living on U as a homework, you can check it. Those who have done the group theory definitely they will be able to do it, so, you can also check please. So, based on that I will assume that $\mathrm{T}^+(n)$ is also group.

So, here $\mathrm{T}^+(n)$is also a group with respect to multiplication. As a result ⟹ $M_1$ & ${M_2}^{-1}$ ∈ $\mathrm{T}^+(n)$ ⟹ $M_1{M_2}^{-1}$ ∈ $\mathrm{T}^+(n)$. So, $M_1{M_2}^{-1}$ ∈ U(n) ⟹ $(M_1{M_2}^{-1})^* = (M_1{M_2}^{-1})^{-1}$ ∈ U(n) as well as $\mathrm{T}^+(n)$.

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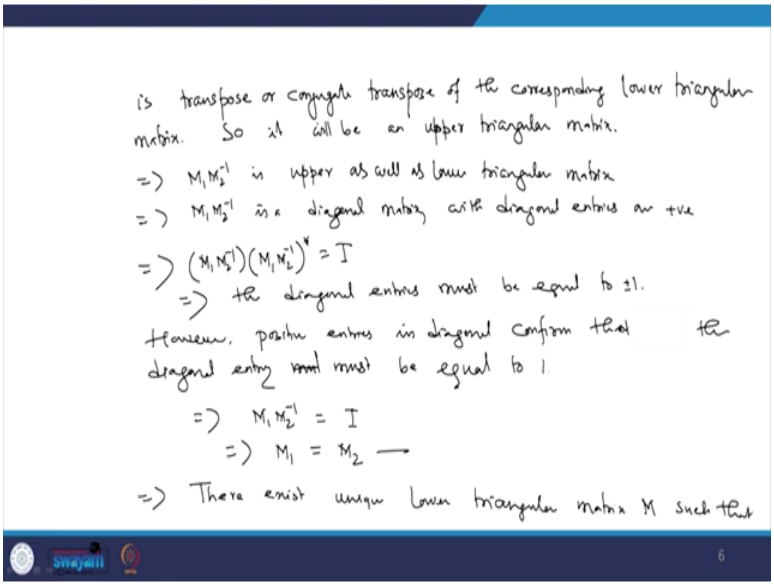


So but we know the, adjoint of lower triangular matrix is transpose or conjugate transpose based on real or complex of the lower triangular matrix. So, it will be an upper triangular triangle matrix $\Longrightarrow M_1{M_2}^{-1}$ is upper as well as lower triangular matrix $\Longrightarrow M_1{M_2}^{-1}$ is a diagonal matrix.

with diagonal entries are positive $\Longrightarrow (M_1{M_2}^{-1})(M_1{M_2}^{-1})^* = \text{I} \Longrightarrow$ the diagonal entries must be $\pm 1$ and however positive entries in diagonal confirm that the diagonal entry must be equal to 1**.** Since, we have seen that $(M_1{M_2}^{-1})(M_1{M_2}^{-1})^*$ =I.

So, this implies that diagonal entry has to be $\pm 1$. But since the diagonal entry positive $\Longrightarrow$ diagonal entry of $M_1{M_2}^{-1} = 1 \Longrightarrow M_1{M_2}^{-1} = \text{I}$. So, this implies $M_1 = M_2 \Longrightarrow$ there exists unique lower triangular matrix M such that U = MA.

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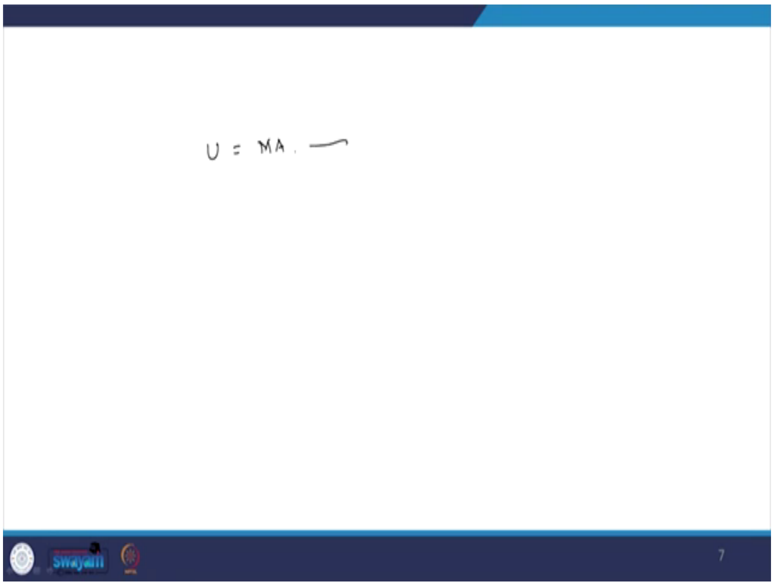


$\Longrightarrow$ The diagonal entry must be equal to $\pm1$ because if you consider that $M_1{M_2}^{-1}$ belongs to the unitary matrix then certainly $M_1{M_2}^{-1} = \mathrm{I}$ $\Longrightarrow$ diagonal entry must be equal to $\pm$ 1 but since the diagonal entry of the each $M_1{M_2}^{-1}$ is positive. So, this says that diagonal entry of $M_1{M_2}^{-1}$ has to be exactly equal to 1 only.

$\Longrightarrow$ $M_1{M_2}^{-1} = \mathrm{I}$. This shows$M_1 = M_2$. So, we see for any nxn complex square invertible matrix will have a unique lower triangular matrix such that what triangular matrix into that given matrix, you will be called to a unitary matrix. This result can be utilized bit more general sense. We will discuss this issue in our next class.