Advanced Linear Algebra
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Lecture – 53
Unitary Operators - I

Welcome to the lecture series on Advanced Linear Algebra. Today we are going to talk about a special operator that is called Unitary Operators. Already we have introduced this terminology in my last lecture while talking about the isomorphism over the inner product spaces. So, let me recall the definition that we made in my last lectures.

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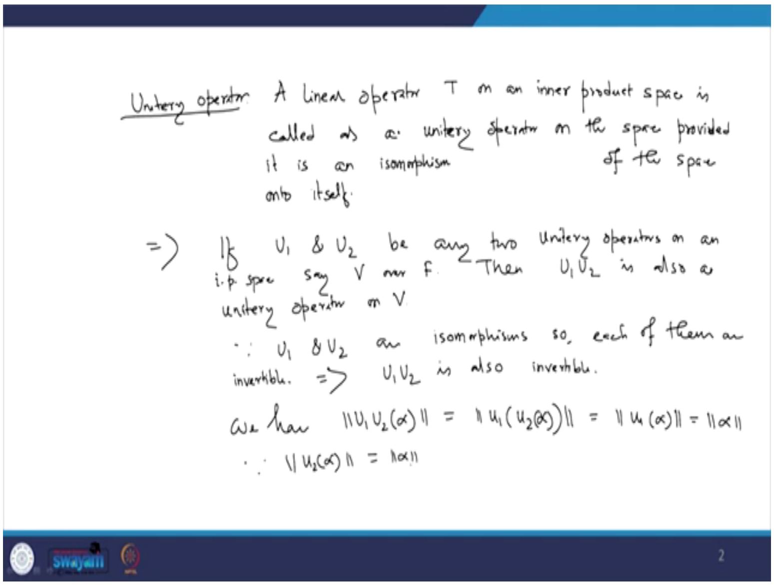

Unitary operators: - A linear operator T on an inner product space is called a unitary operator on the space provided, it is an isomorphism of the space onto itself. So, what I want to show you an unitary operator on an inner product space, a linear operator T such that it is an isomorphism of the space onto itself. That is, it is a 1-1 & onto and it preserves the inner product $\Longrightarrow$ if U_1 and U_2 be any two unitary operators on an inner product space, say V over say, field F.

Then U_1 and U_2 is also a unitary operator on V. How? See since U_1 and U_2 are isomorphism, so, each of them invertible. So, U_1 is inter invertible, U_2 is invertible, so, the product will be also invertible $\Longrightarrow U_1U_2$ is also invertible, into means composition basically. Now, we have to also show that the product U_1 and U_2 preserves inner products.

We see we have $\|U_1U_2(\alpha)\|$ = $\|U_1(U_2(\alpha))\|$ = $\|U_1(\alpha)\|$ = $\|\alpha\|$ because $\|U_2\| = \|\alpha\|$

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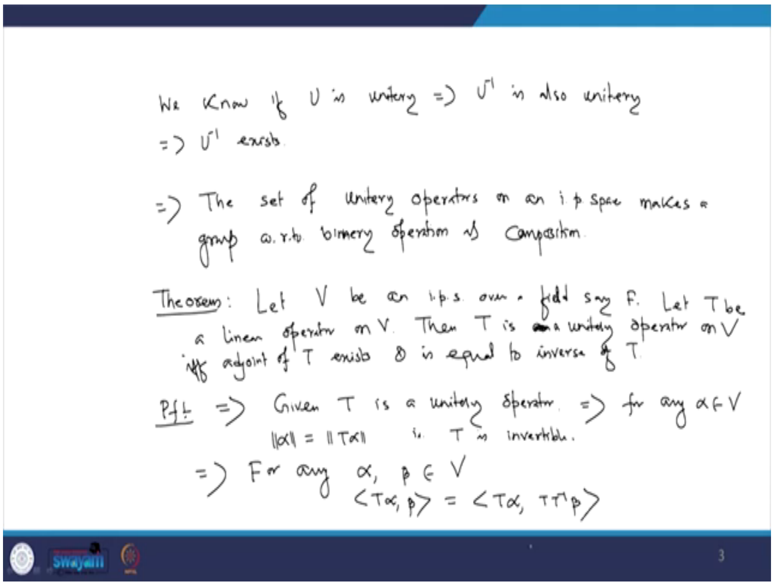

We know, if U is unitary $\Longrightarrow$ U^{-1} is also unitary $\Longrightarrow$ U^{-1} exist $\Longrightarrow$ The set of unitary operators on an inner product space make a group with respect to binary operation as composition. So, say if I consider an inner product space say V. If I collect all the unitary operators, then the set of unitary operators form a group with respect to composition as a binary operation.

Apart from this, we are also curious to know is there any relation between unitary operator and it is corresponding adjoint operator? I mean to say existence of adjoint operator of a unitary operator. So, to answer this question, let us give a small result like this in terms of theorem. Theorem: - Let V be an inner product space(i.p.s.) over a field say F. Let T be a linear operator on V then T is a unitary operator on V if and only if adjoint of T exist and is equal to inverse of T.

So, let me give a quick proof of these small results. See here I am not talking about the V as a finite, dimensional or infinite dimensions. I am taking V is an inner product space. So, let us see whether finite dimensional concept is required or not to prove this result. Proof: - let us first consider, if part given T is a unitary operator means it is an isomorphism which preserves the inner products.

$\Longrightarrow$ for any $\alpha \in$ V, $\|\alpha\| = \|T\alpha\|$ i.e. T is invertible $\Longrightarrow$ for any $\alpha, \beta \in$ V, $< T\alpha, \beta > = < T\alpha, TT^{-1}\beta >$ because $TT^{-1} = I$.

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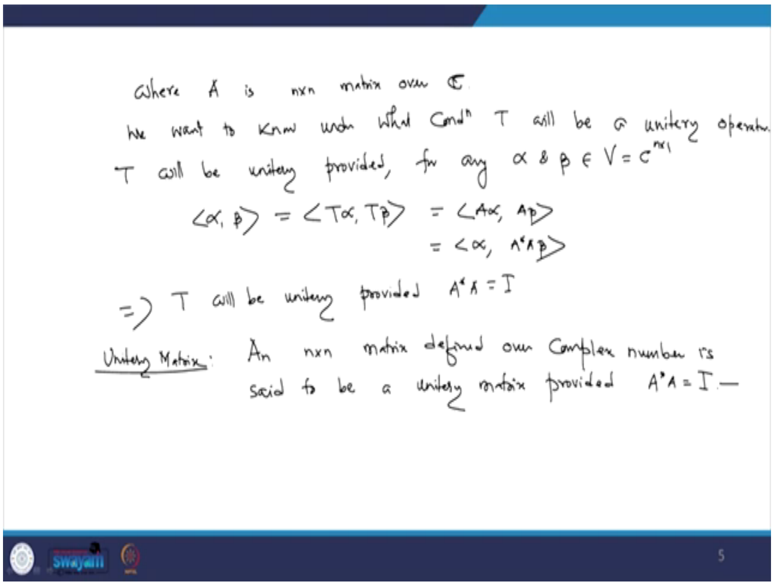

$\Longrightarrow < \mathrm{T}\alpha, \beta > = < \mathrm{T}\alpha, \mathrm{TT}^{-1}\beta >\ = < \alpha, \mathrm{T}^{-1}\beta >$ since T^{-1} is unitary $\Longrightarrow \mathrm{T}^*$ exists & $\mathrm{T}^* = \mathrm{T}^{-1}$ i.e $\mathrm{TT}^{-1} = \mathrm{I}$. Now, let me consider the only if part.

$\Longleftarrow$ Given adjoint of T exist i.e. T^* exist and $\mathrm{T}^*\mathrm{T} = \mathrm{I}$. So, this implies I have to check that T is an automorphism it means T is an isomorphism. So, since T is 1-1& onto, an inverse exist means it is a 1-1 & onto. And so, I can say that it will be in an automorphism provided T preserves inner products. So, for this let me consider for any $\alpha, \beta \in$ V, $< \mathrm{T}\alpha, \mathrm{T}\beta > < \alpha, \mathrm{T}^*\mathrm{T}\beta > = < \alpha, \mathrm{I}\beta > = < \alpha, \beta > \Longrightarrow$ T is unitary operator. $< \mathrm{A}\alpha, \mathrm{A}\beta > = < \alpha, \mathrm{AA}^*\beta >$

Ex:- Let V = C^{nx1} I mean to say that is a column matrix over the complex plane & inner product in V is standard inner product inner product i.e. < X, Y > = $\mathrm{Y}^*\mathrm{X}$. Consider a linear operator T on V defined by TX = AX, where A is nxn matrix over the complex number C.

Now, we want to know under what conditions T will be a unitary operator provided for any $\alpha, \beta \in$ V such that $< \alpha, \beta > = < \mathrm{T}\alpha, \mathrm{T}\beta > = < \mathrm{A}\alpha, \mathrm{A}\beta > = < \alpha, \mathrm{AA}^*\beta >$

$\Longrightarrow$ T will be unitary provided $\mathrm{A}^*\mathrm{A} = \mathrm{I}$. See we have defined the unitary operators. Now, I can also define unitary matrix. Unitary matrix :- An nxn matrix define over complex number is said to be be a unitary matrix provided $\mathrm{AA}^* = \mathrm{I}$. Here in my definition I have fixed $\mathrm{AA}^* = \mathrm{I}$.

I mean I have seen identity only in one side. If I go AA^* = I? That is the question, I will answer this question soon but right now I will say that it will be $\mathrm{A}^*\mathrm{A}$ = I. I will show you later on.

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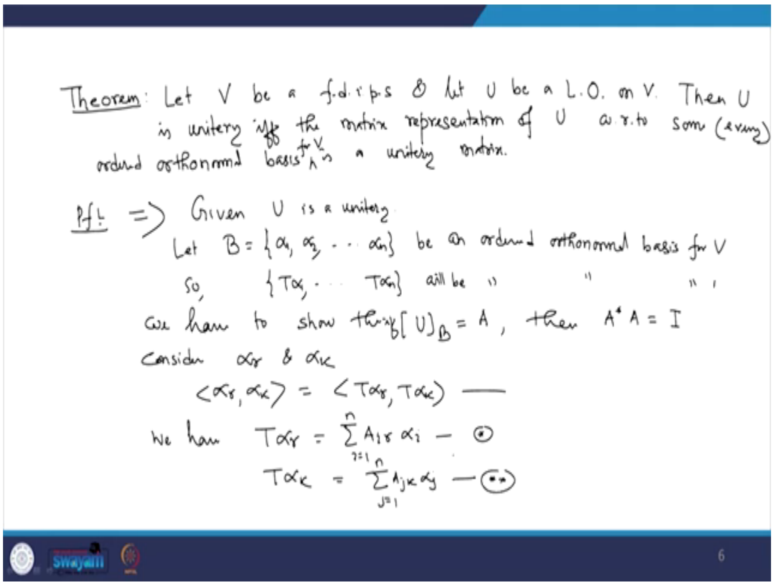

So, to answer these questions, so, let me give a small result it is like this. So, let us talk about one nice results it is like this, Theorem: - V be a finite dimensional inner product space, let U be a linear operator on V. Then U is unitary if and only if, the matrix representation of U with respect to some(every) or orthonormal basis for V is a unitary matrix and mention one thing.

I have told you that is to inner product spaces we have defined some equivalent statement when the inner product spaces is are finite dimensions. So, here also, I will say unitary operator over a finite dimensional inner product space also satisfied all that all sort of equivalent statement. I mean to say if U is a unitary operator then it preserves inner products implies it satisfy the criteria that it carries every orthonormal basis to orthonormal basis for the space or some orthonormal basis for V onto a orthonormal basis of the V.

So, based on that only I have written here, you will be unitary if and only if, the matrix representation of U with respect to some(every) order orthonormal basis for V is a unitary matrix. So, let us give a proof. Proof:- (i) $\Longrightarrow$ Given U is unitary. Let B = $\{\alpha_1, \alpha_2, \ldots\ \alpha_n\}$ be an order orthonormal basis for V. So, $\{\mathrm{T}\alpha_1, \mathrm{T}\alpha_2, \ldots\ \mathrm{T}\alpha_n\}$ will be an order orthonormal basis for also V.

So, we have to show $[U]_B$= A, where A will be an nxn matrix then $\mathrm{A}^*\mathrm{A} = \mathrm{I}$, we have to prove this one identity matrix. Consider α_r & α_k so, we have $<\alpha_r, \alpha_k> = <\ \mathrm{T}\alpha_r, \mathrm{T}\alpha_k>$

from the definition of the matrix representations. We have T$\alpha_r = \sum_{i=1}^{n} A_{ir}\alpha_i$. This is from the definition only we, I hope you remember this definition. And similarly, T$\alpha_k = \sum_{j=1}^{k} A_{jk}\alpha_j$

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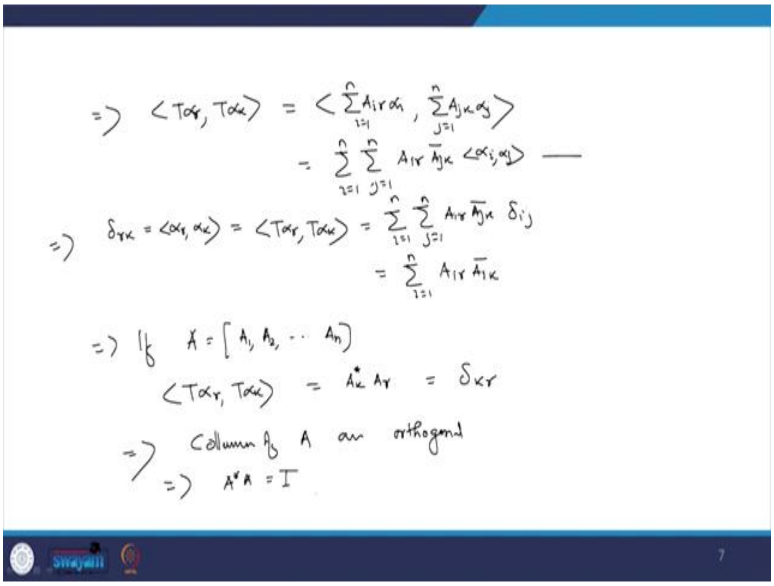

$\Longrightarrow$ $<$ Tα_r, Tα_k $> = < \sum_{i=1}^{n} A_{ir}\alpha_i , \sum_{j=1}^{n} A_{jk}\alpha_j$ $=$ $\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ir}\,\overline{A_{jk}} < \alpha_i, \alpha_j >$ $\Longrightarrow$ $\delta_{rk} = < \alpha_r, \alpha_k > = < T\alpha_r, T\alpha_k > =$ $\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ir}\,\overline{A_{jk}}\,\delta_{ij}$ $=$ $\sum_{i=1}^{n} A_{ir}\,\overline{A_{ik}}$ So,

$\Longrightarrow$ if the matrix A = $[A_1, A_2, \ldots A_n]$ then I have $<$ Tα_r, Tα_k $>$ = $A^*{}_k A_r = \delta_{kr}$ $\Longrightarrow$ columns of A are orthogonal $\Longrightarrow$ $A^*A = I$.

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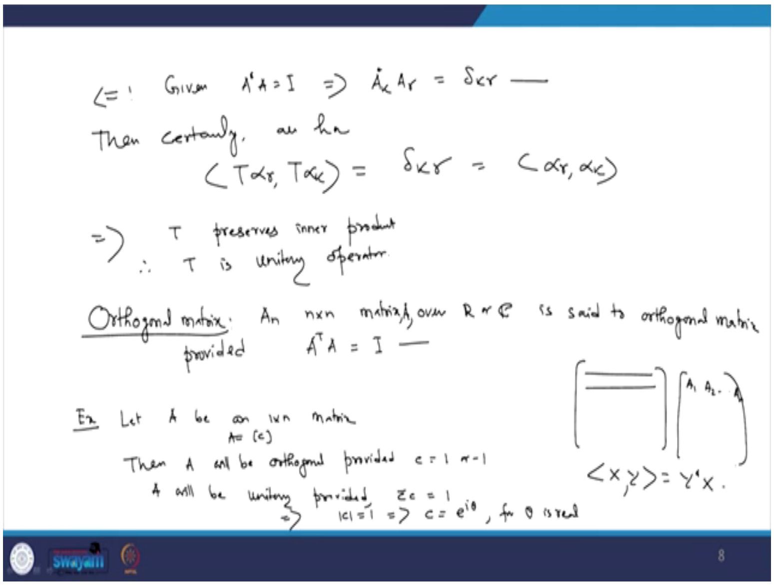

Now, conversely, (only if) $\Leftarrow$ Given that given $A^*A = I \Longrightarrow A^*{}_k A_r = \delta_{kr}$ it is given to us, so, this is given like this. Then certainly, we have $< T\alpha_r, T\alpha_k > = \delta_{kr} = < \alpha_r, \alpha_k > \Longrightarrow$ T preserves inner product, so, T is unitary operator.

Now, based on this, let me define another nice matrix that is called orthogonal matrix. Orthogonal matrix:- An nxn matrix over real or complex is said to be is said to be orthogonal matrix provided, $A^T A = I$. Let me also recall one thing here on my while proving the last results that is if $A^*A = I$ then it will be give me unitary operators.

Here $A^*A = I$ means we are saying that if I write in a explicit matrix form then you see that the column see this is basically. This is $\begin{bmatrix} \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \\ \dots \dots \dots \dots \dots \dots \dots \end{bmatrix} \begin{bmatrix} A_1 & A_2 & \dots & A_n \\ & & & \\ & & & \end{bmatrix}$ This column will be this row and each column will go to this one row and with the conjugate bar. So, in that case we see that with respect to standard inner product, if I consider over the space $< X, Y > = Y^*X$ then each column of the matrix A.

I mean the set of column matrix of A are orthogonals and the row matrices of A are also orthogonals. So, it will be irrelevant I mean if I instead, if I take instead of A^*A even if I take $AA^* = I$, then also I will have identity matrix. So, one sided proof implies two sided proof here. I mean I have shown one-sided proof of the existence of the inverse but since the nature of the matrix a such that we can see that even if I go for the other side.

I will also have the same result. Therefore, in my definition of the orthogonal matrix, I am saying that if the $A^T A = I$ then I will say that it is a orthogonal matrix by default. It is also satisfying that $AA^T = I$ A, so, do not confuse with this one that is it only one sided always. I have to write $A^T A$ or I can also write it AA^T this confusion should not be there.

And this confusion should not be there because of this principle of this concept of the orthogonal matrix here. So, let us see some example of orthogonal matrix as well as unitary matrixes. Let A be 1x1 matrix i.e. A = [c]. Then A will be orthogonal provided, $|c| = \pm 1$ because if c = 1 then this transpose will also same and so, 1x1 = 1 i.e. 1x1 identity matrix.

And similar, if c = –1 then –1x –1 = 1. So that is why, if you consider 1x1 matrix A and it is written like c then A will be orthogonal provided c = 1 or c = –1. Now, A will be unitary suppose you want to check whether it is unitary or not unitary, provided what will the condition for unitary? Provides c $\bar{c}$ = 1, so, this implies |c| = 1. So, this implies c = $e^{i\theta}$ where θ is a real number real, so that |c| = 1.

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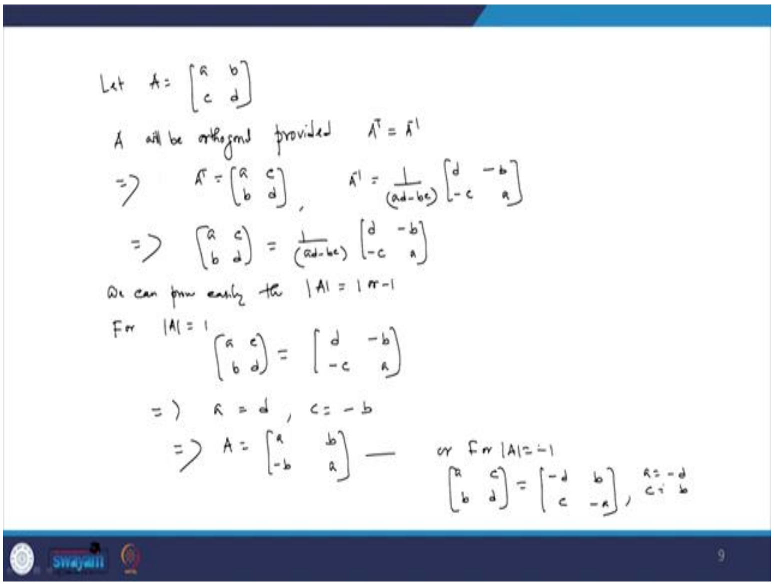

Now, let me consider let A = $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Now, let us check under what condition the matrix A will be orthogonal. A will be orthogonal, provided, $A^T = A^{-1}$. Let us quickly find out what is A^T? What is A^{-1}? $\Rightarrow A^T = A^{-1}$? $A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$ and $A^{-1} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \frac{1}{(ad-bc)}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. Now, quickly we can check, we can prove easily, |A| = $\pm$ 1. If you consider orthogonal matrix |A| = $\pm$ 1. So, for |A| = 1, I will have the condition that $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\Rightarrow$ a = d, c = –b $\Rightarrow$ A = $\begin{bmatrix} a & c \\ b & d \end{bmatrix}$, so, this is one possible. If the A equal to like this then I can immediately say that A will be orthogonal & for |A| = – 1 $\Rightarrow$ $\begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix}$,I will have a = –d and c = b.

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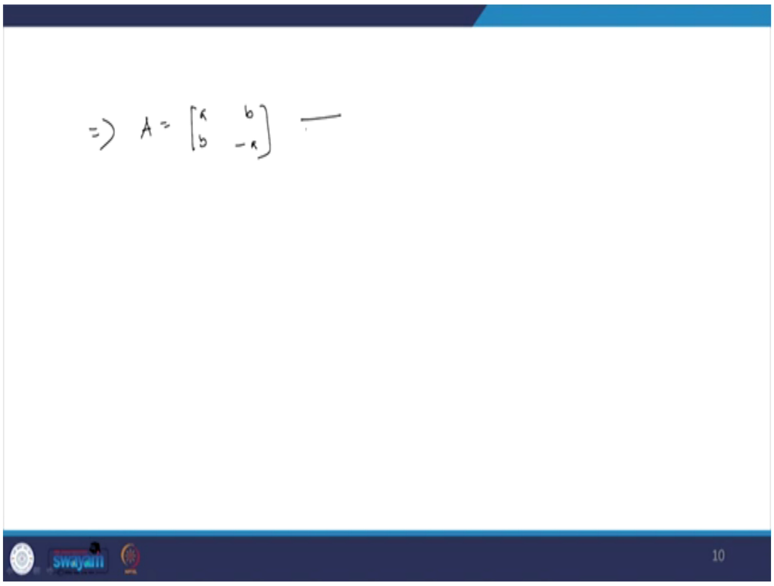

$\Longrightarrow$ A = $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$. So, the matrix will be like this. There are two possible structure for the A for which you will have these 2x2 matrix will be orthogonal. We will also quickly cross check under what conditions it will be a unitary matrix using exactly same analysis. You can do it so, continue in our next class together.