Advanced Linear Algebra
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Lecture – 52
Isomorphism in Inner Product Spaces

Welcome to the lecture series on Advanced Linear Algebra. Today we are going to talk about an isomorphism in inner product space. Already, we have defined the meaning of isomorphism in vector spaces. Now, isomorphism in an inner product space is defined like this.

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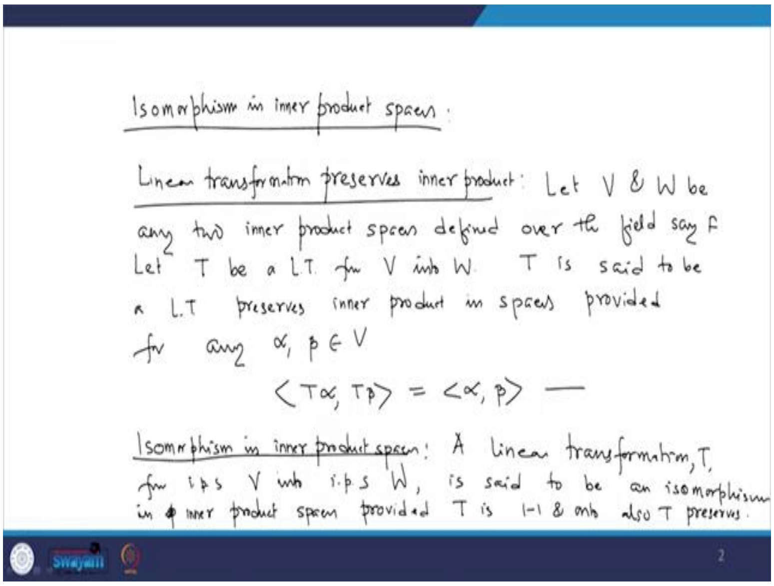

So, isomorphism in inner product spaces, see isomorphism in vector spaces we have defined is a linear transformation from V to W. Here V and W, are defined over the same field and if the linear term is 1-1, onto then used to say that T is the isomorphism from V on to W. Now, in the case of the isomorphism inner product spaces, we are looking not only that T will be 1-1, onto. It has to also preserves the inner products in the spaces.

So, what is the meaning of a linear transformations preserves inner product in the corresponding spaces? So, linear transformation preserves inner product. What is the meaning of this line? Let V and W be any two inner product spaces define over the field, say F. And both the inner product spaces are defined over the same field. Let T be a linear transformation from V into W.

T is said to be a linear transformations preserves inner product in spaces provided for any $\alpha, \beta \in$ V, $< T\alpha, T\beta > \ = < \alpha, \beta >$ See note that the $< T\alpha, T\beta >$ it is an inner product in W whereas $< \alpha, \beta >$ is in V. So, if this property holds good then I will say that linear transformations preserves the inner products space.

In the case of in isomorphism inner product spaces, I am saying that it is a linear transformation. So, let me define isomorphism in inner product spaces. A linear transformation T: V → W where both the vector spaces are inner product spaces are defined about the same field is said to be an isomorphism in inner product spaces provided T is 1-1 and onto also T preserves inner product.

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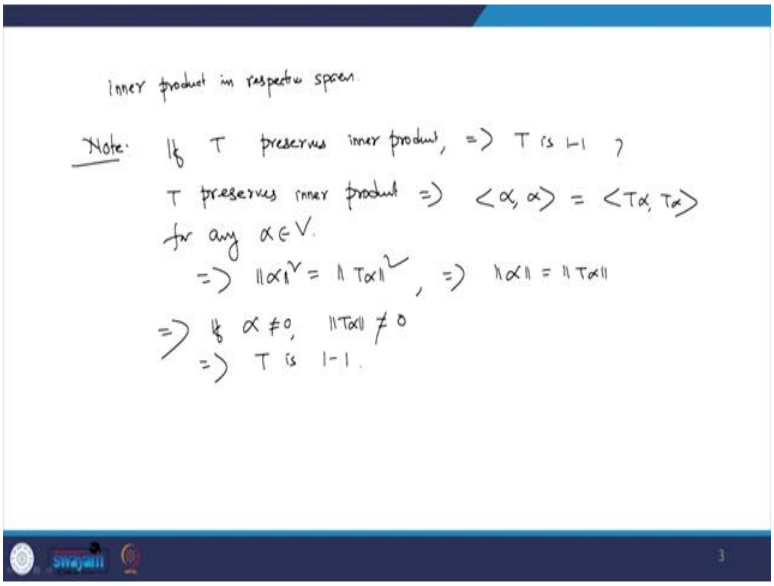

Inner product in respective spaces so that the meaning of the isomorphism in an inner product spaces. See if I say that if T preserves inner product then this implies that T is 1-1 how? T preserves inner product $\Longrightarrow < \alpha, \alpha > = < T\alpha, T\alpha >$ for any $\alpha, \beta \in$ V $\Longrightarrow \ ||\,\alpha||^2 = \ ||\,T\alpha||^2$

Here norm is induce by the inner product space $\Longrightarrow ||\,\alpha|| = ||\,T\alpha|| \Longrightarrow$ if $\alpha \neq 0, ||\,T\alpha|| \neq 0$ $\Longrightarrow T$ is 1-1, so, T is a isomorphism in an inner product V on to inner product W. It can be defined like this. It is a T: V → W which satisfies preserving of the inner products.

I mean to set the preserves inner product in this places, also so, 1-1, no need to say because 1-1 is coming as a consequence of the that preserving inner product implies 1-1. But it is also not wrong if I say that a 1-1, onto linear transformations between the two inner product which is V and W will be isomorphism provided it preserves the inner product in the spaces.

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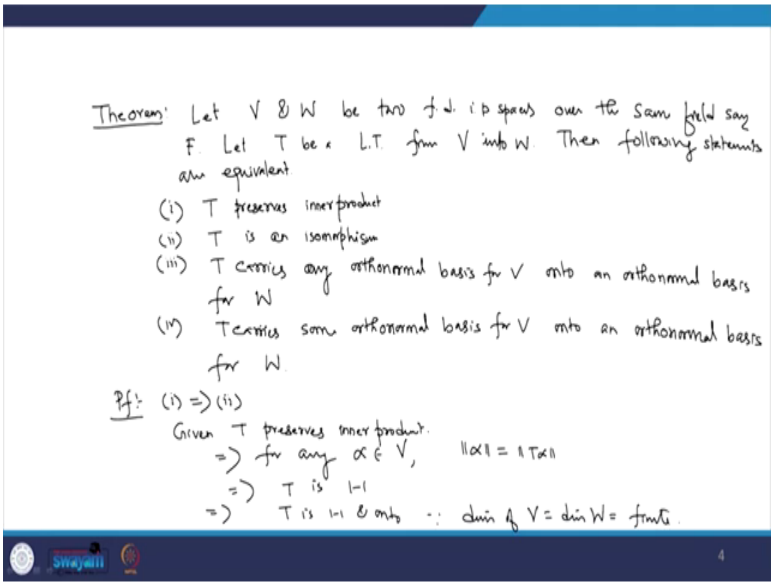

So, one can also redefine isomorphism, inner product spaces, and some equivalent statements. So, let me talk about that one. That is, I am talking about the inner product spaces when the spaces have finite dimensions. So, let me talk in terms of this theorem it is saying like this. Theorem: - Let V and W be two finite-dimensional inner product spaces over the same field, say F.

Let T be a linear transformation from V into W. Then, the following statements are equivalent. (i) T preserves the inner product (ii) T is an isomorphism. See, sometimes you may confuse that isomorphism. We know if it is 1-1 & onto between the two spaces. But in the case of inner product spaces I have defined isomorphism as 1-1, onto linear transformation, as well as the transformation preserves the inner product.

Sometimes some books will say that it is written at inner product spaces isomorphism just to differentiate between the isomorphism over the two vector spaces and isomorphism over the two inner product spaces. But here, I am using simple isomorphism because I have considered this is isomorphism over the inner product spaces only. So, if T preserves in inner product, then it also implies that T is an isomorphism.

(iii) T carries any orthonormal basis for V onto an orthonormal basis for W (iv) T carries some orthonormal basis for V onto an orthonormal basis for W. So, let us quickly prove these theorems I want to show that if a T is a linear transformation from V into W, where V & W are finite-dimensional inner product spaces defined over the same field.

V and W have the same dimensions in this case (iv). And (i) the T preserves an inner product, and (ii) T is an isomorphism. (iii) T carries any orthogonal basis for V onto an orthogonal basis for W (iv) T carries some orthogonal basis for V onto an orthogonal basis for W. This all four statements are equivalent, so let us quickly prove that one.

So, let us prove first in how (i) $\Longrightarrow$ (ii)? So, given T preserves inner product $\Longrightarrow$ for any $\alpha \in$ V, $\|\alpha|| = ||T\alpha|| \Longrightarrow$ T is 1-1. So, T it is given to us that T preserves inner product and T is also 1-1 and since the dimension of V & W are finite and same. So, it is also onto $\Longrightarrow$ T is 1-1 & onto since dimension of V = dimension of W = finite. So that is why T is 1-1 & onto and preserves the isomorphism.

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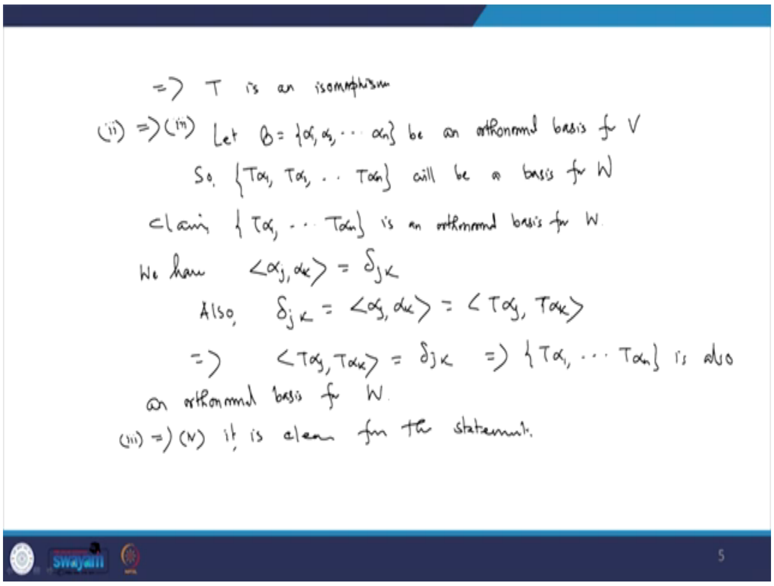

$\Longrightarrow$ T is an isomorphism, now, let us prove how does (ii) $\Longrightarrow$ (iii) given to us that T is an isomorphism. So, it implies that it preserves also in our inner products I have to show that it carries an orthonormal basis for V onto an orthonormal basis for W. So, let B = $\{\alpha_1, \alpha_2, \dots\ \alpha_n\}$ be an orthonormal basis for V. So, $\{T\alpha_1, T\alpha_2, \dots\ T\alpha_n\}$ will be a basis for W because since T is 1-1.

So, we know that T carries linear independent set to linear independent set. So, on the basis of that I can say that B is an orthogonal basis, then $\{T\alpha_1, T\alpha_2, \dots\ T\alpha_n\}$ has to be basis for W. But my claim is that $\{T\alpha_1, T\alpha_2, \dots\ T\alpha_n\}$ in orthonormal basis for W. Claim $\{T\alpha_1, T\alpha_2, \dots\ T\alpha_n\}$ is an orthonormal basis for W we have, $< \alpha_j, \alpha_k >$ = δ_{ij}

And we know also $\delta_{ij} = < \alpha_j, \alpha_k > = < T\alpha_j, T\alpha_k > \Longrightarrow \; < T\alpha_j, T\alpha_k > = \delta_{ij}$ = 1 or 0 when i = j and i $\neq j \; \Longrightarrow$ {Tα_1, Tα_2, ... Tα_n}is also orthonormal for W. Now (iii) $\Longrightarrow$ (iv) It is very straightforward because (iii) is a bigger hypothesis.

It carries, I mean saying that T carries each orthonormal basis is to orthonormal basis for the W. So, certainly it will also include some orthonormal basis for the V to also some orthonormal basis for the W. So, it is clear from the statement.

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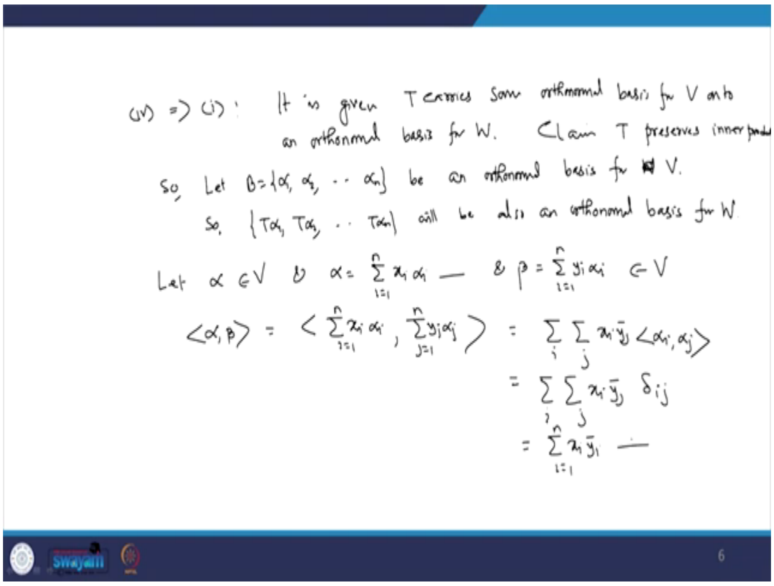

So now, I want to show (iv) $\Longrightarrow$ (i) It is given T carries some orthonormal basis for V on to an orthonormal basis for W. So, let so, I have now I want to show that T preserves the inner product. That is basically (i), claim T preserves inner product, so, let me consider B = $\{\alpha_1, \alpha_2, \ldots \; \alpha_n\}$ be an orthonormal basis for V. So, this is the alternate basis for V. So, according to given conditions {Tα_1, Tα_2, ... Tα_n}will be also an orthonormal basis for W.

I have to say that for any two vectors $\alpha, \beta \in$ V, $\alpha = \sum_{i=1}^{n} x_i\alpha_i$, $\beta = \sum_{i=1}^{n} y_i\alpha_i$ such that $< \alpha, \beta > = \sum_{i=1}^{n} x_i\alpha_i, \sum_{j=1}^{n} y_j\alpha_j > \; = \; \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \, \overline{y_j} < \alpha_i, \alpha_j > \; = \; \sum_{i=1}^{n}\sum_{j=1}^{n} x_i \, \overline{y_j} \;\; \delta_{ij} = \sum_{i=1}^{n} x_i \, \overline{y_j}$. Now, let us see what about the $< T\alpha, T\beta >$

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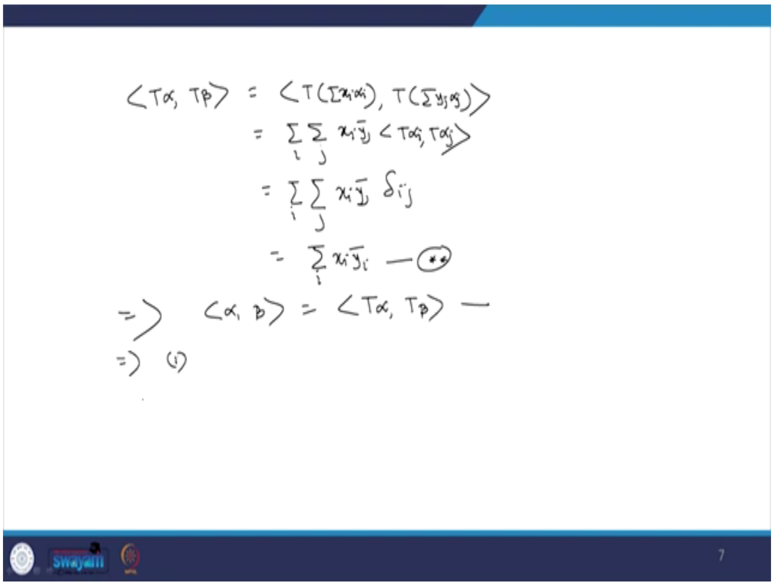

So, $< T\alpha, T\beta > =< T\{\sum_{j=1}^{n} x_i\alpha_i\}, T\{\sum_{j=1}^{n} y_j\alpha_j\} > = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\, \bar{y}_j < T\alpha_i, T\alpha_j > = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\, \bar{y}_j\ \delta_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_i\, \bar{y}_j \ \Rightarrow < \alpha, \beta > = < T\alpha, T\beta > \Rightarrow$ (i)

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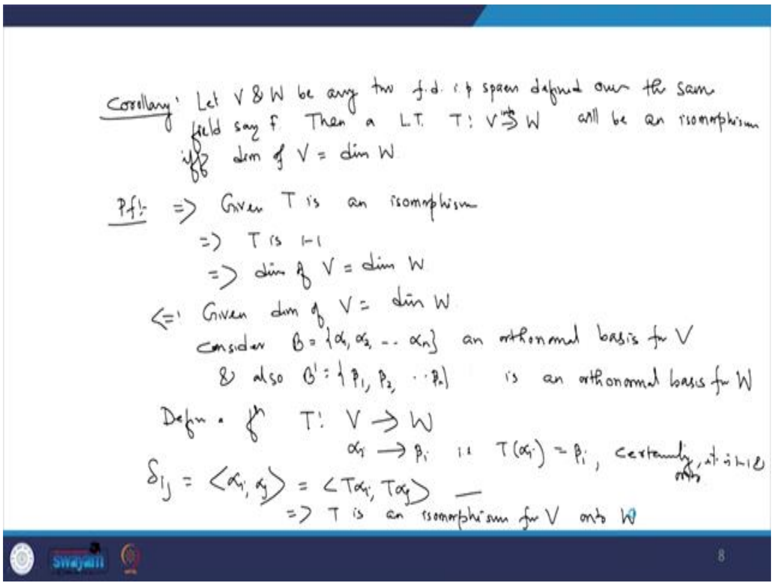

Corollary: - Let V and W be any two finite dimensional inner product spaces defined over the same field say F. Then a linear transformation T: V → W will be an isomorphism if and only if dimension of V = dimension of W. The proof is very straight forwards we can consider as homework also. Let me talk very quickly to see if it is an isomorphism. Then certainly, it has to be 1-1.

So, this means that the dimension of V has to be equal to the dimension of W. Now, let me consider if the part is very straightforward. Proof: - Given T is an isomorphism ⟹ T is 1-1 ⟹

dimensions of V = dimension of W because isomorphism 1-1, as well as onto. So, dimension of V = dimension W. Now, other way given dimension of V = dimension of W.

So, I have to show that there exists a linear transformation from V into W which is 1-1, onto and preserves the inner products. Then only I can say that linear transformation is an isomorphism from V on to W. So, what I will do is consider B = $\{\alpha_1, \alpha_2, \dots \alpha_n\}$ an orthonormal basis for V. And also B′ = $(\beta_1, \beta_2, \dots \beta_n)$ an orthonormal basis for W.

This is possible because both of them have the same dimension and are finite, so, using the Gram-Schmidt orthonormal procedure, one can have a corresponding orthonormal basis. So now, let me define so, define a function T: V $\rightarrow$ W which is mapping $\alpha_i \rightarrow \beta_i$ i.e. T$(\alpha_i) = \beta_i$ so, $< \delta_{ij} = < \alpha_i, \alpha_j > = \; < T\alpha_i, T\alpha_j > \implies$ T is an isomorphism from V onto W.

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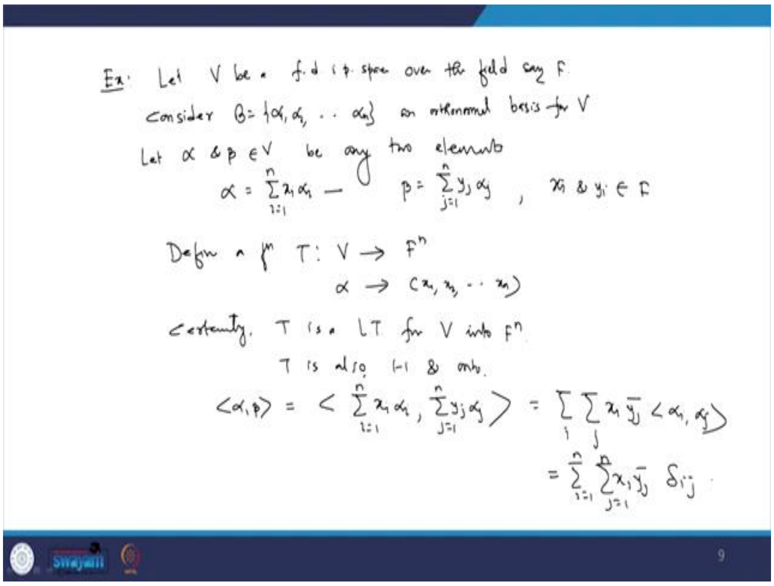

Let me consider a couple of examples of isomorphism so that you can visualize this concept. So, let me take first very straightforward examples. It is like this let V be a finite dimensional inner product space over the field F. Consider B =$\{\alpha_1, \alpha_2, \dots \alpha_n\}$ an orthonormal basis for V. So, I will show that for consideration of each orthonormal basis for B will introduce an isomorphism between V onto F^n.

I mean F^n the space of all n dimensional array and I will consider the inner product as standard inner product on F^n. So, I have taken B equal to like this in orthonormal basis for V. Let $\alpha, \beta \in$ V, $\alpha = \sum_{i=1}^{n} x_i \alpha_i$, $\beta = \sum_{i=1}^{n} y_j \alpha_j$ & $x_i, y_j \in$ F. Now, if I defined a function T: V $\rightarrow$ F^n where

F^n is taken as the space of all n-tuple spaces, and of course, it is also in our product where the inner product is in F^n.

$\alpha \rightarrow (x_1, x_2, \ldots x_n)$. So then, certainly, T is a linear transformation from V into F^n. Already we have checked, and you will see that it is also 1-1, and onto. Onto is coming in the consequence of the finite dimension. So, 1-1 implies onto also here. Here I will say that $< \alpha, \beta > = \sum_{i=1}^{n} x_i \alpha_i , \sum_{j=1}^{n} y_j \alpha_j > = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \, \overline{y_j} < \alpha_i, \alpha_j > = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i \, \overline{y_j} \, \delta_{ij}$

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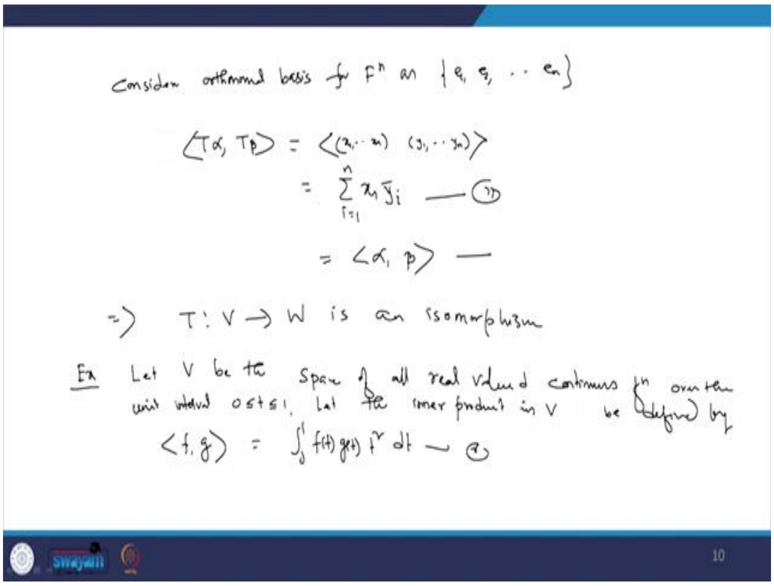

Now, I will be considering orthonormal basis this is for F^n as $\{e_1, e_2, \ldots e_n\}$ where our standard basis in the F^n i.e. $e_1 = (1,0, \ldots 0), e_2 = (0,1, \ldots 0) \ldots e_i = (0, \ldots 1, \ldots 0)$. So, in that case I have $< T\alpha, T\beta > = < (x_1, x_2, \ldots x_n), (y_1, y_2, \ldots y_n) > = \sum_{i=1}^{n} x_i \, \overline{y_j} = < \alpha, \beta >$

$\Longrightarrow$ T from V into W is an isomorphism. Ex: - Where I need to show that a linear transformation between two inner product spaces may preserve the inner products without being an isomorphism. So, let me construct this. Let V be the space of all real-valued continuous functions over the unit interval $0 \le t \le 1$.

Let the inner product in V we defined by $< f, g > = \int_0^1 f(t) g(t) \, t^2 \, dt$

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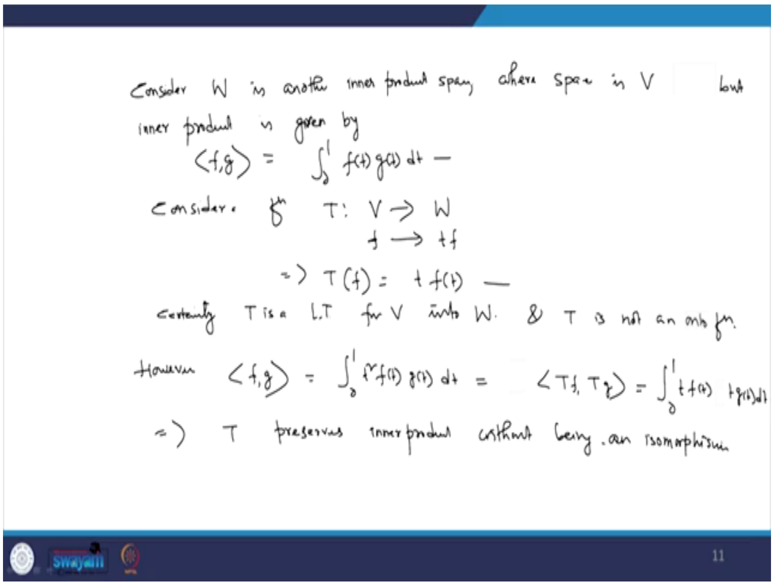

Let me consider W is another inner product space is V but inner product is given by given by like this $< f, g > = \int_0^1 f(t)g(t)\, dt$. This is my inner product in W. I mean to say W is also space of all real valued continuous function defined over the unit interval $0 \leq t \leq 1$. But equal to the inner product which is I can say, standard inner product over that space.

Consider a function T: V → W such that f → tf ⟹ T(f) = tf(t). So, this is a definition, then certainly, T is a linear transformation from V into W, and T is not a onto function it cannot be isomorphism. But let us see, however, if I see that $< f, g > = \int_0^1 \mathrm{t}^2 f(t)g(t)\, dt = < Tf, Tg >$ $\int_0^1 tf(t)\, tg(t)\, dt$

⟹ T preserves inner product without being an isomorphism. So, simply preserving inner product does not imply that the corresponding linear map will be an isomorphism between two inner product spaces. That is why, if a first theorems, I use the word that V and W, are two finite dimensional inner product spaces.

So that result is valid only if V and W, are finite dimensions. I will show you how this interesting concept introduces an interesting operator in this inner product space, which operator plays a key role in understanding linear algebra, which we will discuss in the next class.