Advanced Linear Algebra
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Lecture – 51
Linear Functionals and Adjoints - IV

Welcome to lecture series of Advanced Linear Algebra. Today, we will discuss about what is adjoint of a linear operator. We have already seen over a finite dimensional inner product space. If you consider any linear operator on it, it is adjoints exist. But what about the, if the dimension of the space is not finite what can you say about this issue? So, I will say that in general it may not exist also.

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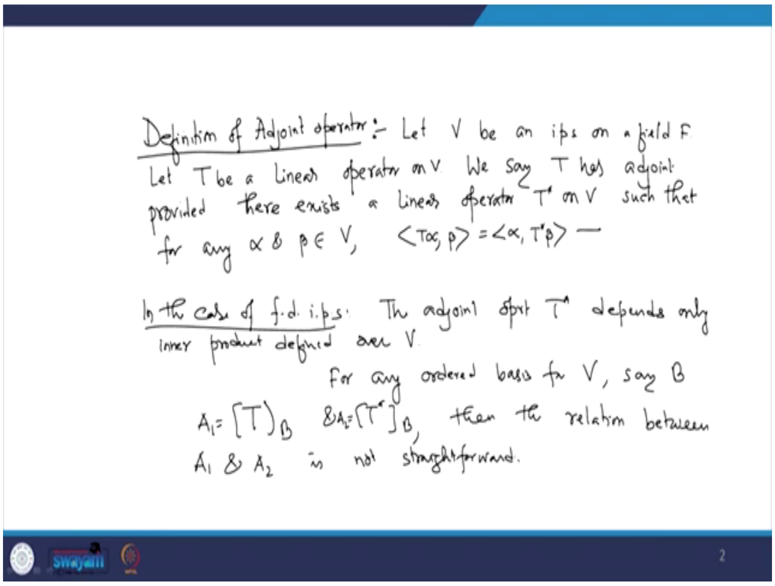

So, let us define, what is adjoint of a linear operator? Let V be an inner product space(i.p.s.) on a field F it may be real or complex. Let T be a linear operator on V. We say, T has adjoint provided there exist a linear operator T^*on V. Such that for any $\alpha, \beta \in$ V, $<T\alpha, \beta> = <\alpha, T^*\beta>$ So, this is the definition of the adjoint operator.

in the case of finite dimensional inner product space, this adjoint operator for every linear operator on it exist. I mean, if you define it, if you take a linear operator on V which is a finite dimensional inner product space. Then they are just a unique adjoint of the operator T. So, in this case, I mean in the case of finite dimension space the adjoint operator T^* depends on inner product defined over your space V.

So, if you change the inner product then the adjoint operator will be also changed. Second things if you consider, for any order basis for V, say B. $[T]_B$ & $[T^*]_B$, the relation between these two is not straight forward. Suppose $A_1 = [T]_B$ & $A_2 = [T^*]_B$ then the relation between A_1 and A_2 is not straight forward.

However, if I consider orthonormal analysis then we know the how they are related. So, in that case, $A_{1_{jk}} = \overline{A_{2_{Jk}}}$ So, this thing we already said.

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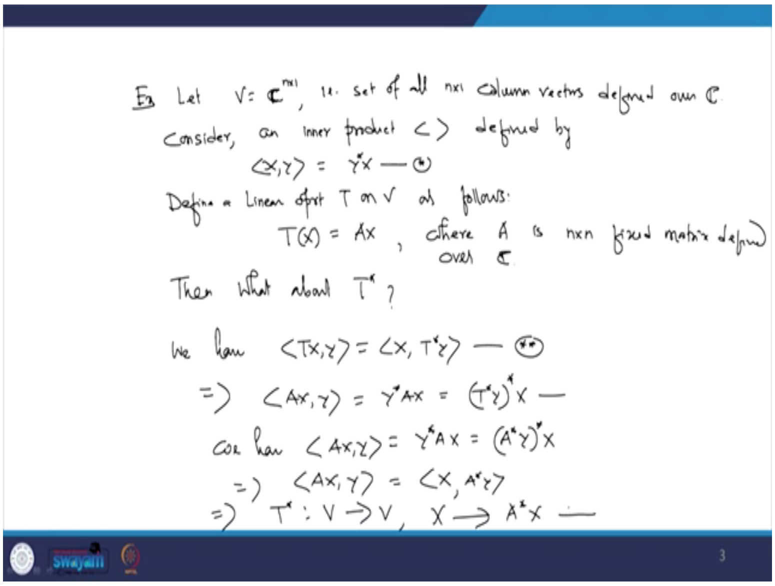

Let me consider couple of examples in different spaces. So, let me consider first, is let V = $\mathbb{C}^{nx1}$ column vectors define over the complex number $\mathbb{C}$, i.e. set of all nx1 column vectors defined over the complex number $\mathbb{C}$. Consider, $< \ >$ defined by $< X, Y > = Y^*X$

So, this is the definition of the inner product then define a linear operator T on V as follows T(X) = AX, where A is nxn fixed matrix define over $\mathbb{C}$. Then what about T^*, I mean to say how the T^* is defined over the space. Certainly, T^* will exist because V the finite dimensional inner product space, so, certainly it is T^* will exist. But how T^* is defined that we have to see?

We have according to the definitions for any X and Y, $< TX, Y > = < X, T^*Y > \implies < AX, Y > = Y^*AX = (T^*Y)^*X$. So, we have to know, what is T^*? We have, $< AX, Y > = Y^*AX = (A^*Y)^*X \implies < AX, Y > = < X, A^*Y >$

$\Longrightarrow$ T*: $V \rightarrow V,\ X \rightarrow$ A*X. So, this is the definition of the adjoint operator. For this vector space when the space is equipped with this type of inner product.

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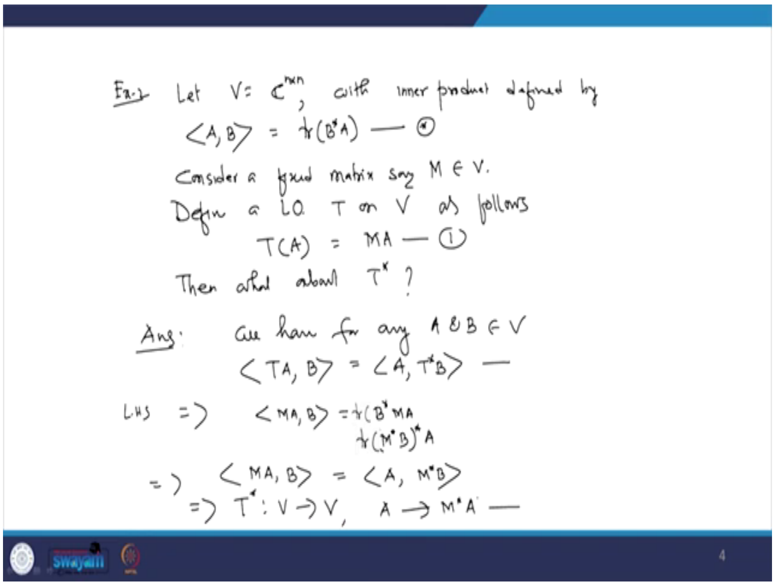

Now, let me just modify and second examples where basically similar to my first examples. So, let me consider let V = $\mathbb{C}^{nxn}$, I mean to say, set of all nxn matrix defined over the complex field $\mathbb{C}$ with inner product defined by for any, < A, B > = tr(B^*A). So, this is my definition of the inner product for this space.

So, consider a fixed matrix, say M ∈ V they defined a linear operator as follows, T(A) = MA. So, this is my definition of the linear operator T on V. Again, since the space is finite dimensional. So, certainly will have adjoint operator of T. Now, the question is how this adjoint operator is defined for this operator. Then what about T^*?

We have, for any A & B ∈ V, < TA, B > = < A, T^*B >, so, this is the definition of the adjoint operator. So, let me define how this T^* is defined over the V. L.H.S. $\Longrightarrow < MA, B > =$ tr$(B^*$MA$) =$ tr$(M^*B)^*$A $\Longrightarrow < MA, B > = < A, M^*B > \Longrightarrow$ T*: $V \rightarrow V,\ A \rightarrow$ M*A, this is definition of the adjoint operator for this problem.

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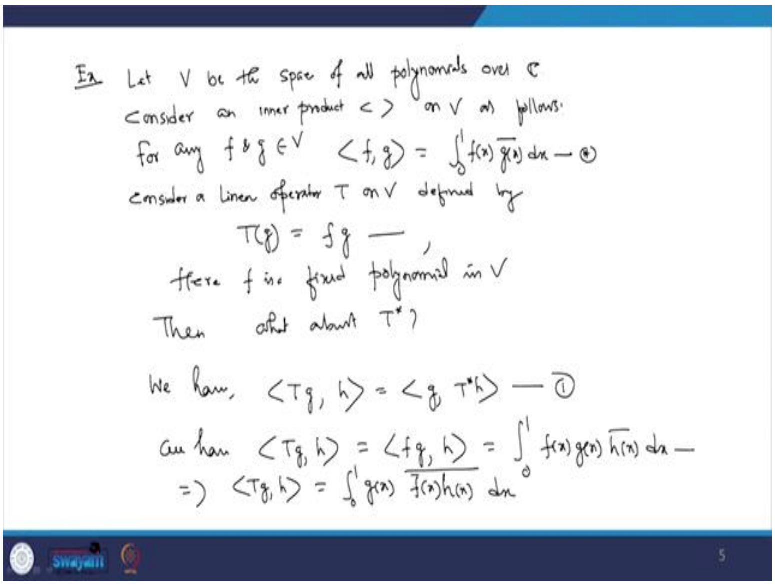

Now, let me take an example over the infinite dimensional space and how to find the adjoint operator for that space? Ex: - Let V be the space of all polynomials over ℂ, I mean to say the coefficient of the polynomials are complex quantity. Consider an inner product on V, as follows for any f & g ∈ V. So, f and g both are the polynomials over the ℂ, < f, g > = $\int_0^1 f(x)\,\overline{g(x)}\,dx$

So, the inner product is defined like this. Consider a linear operator T on V defined by T(g) = fg, I mean f is a polynomial and g is also polynomial. Here, f is a fixed polynomial in V, I mean to say **(()) (15:39)** randomly one polynomial, say f. The defining operator T on V as mapping T(f) = fg. So now, the question is what will be the corresponding T^*? What about T^*?

We have, h & g ∈ V such that < Tg, h > = < g, T^*h > so, you are looking for the adjoint operator T which satisfy these conditions. We are not saying that here T^* will exist if exist then how this T^* is defined? That we have to see it. If this means they have to satisfied this type of condition. We have, < Tg, h > = < fg, h > = $\int_0^1 f(x)g(x)\,\overline{h(x)}\,dx \implies$ < Tg, h > = $\int_0^1 g(x)\,\overline{\overline{f(x)}\,h(x)}\,dx$

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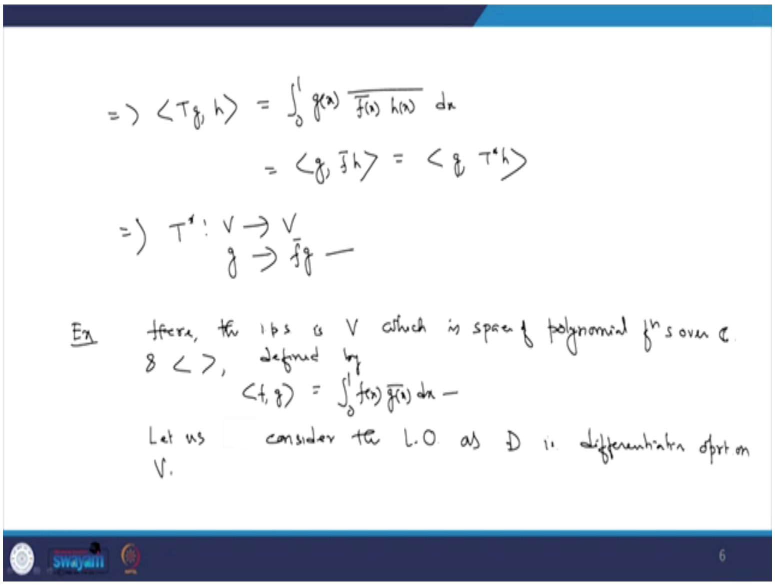

$\Longrightarrow$ < Tg, h > = $\int_0^1 g(x)\, \overline{\overline{f(x)}\, h(x)}\, dx = < g, \bar{f}h > = < g, T^{*}h >$ since $\bar{\bar{f}}$ means that is f itself. T*: $V \to V,\ g \to \bar{f}g$, this is the definitions, so, here we have seen, even though the space is infinite dimensional.

But for a given linear operator like this, we have found a adjoint operator of it. Now, let me slightly change the operators. So, space is same here the inner product space is be a V which is space of polynomial functions over the complex number ℂ and same inner product and < , > defined by < f, g > = $\int_0^1 f(x)\ \overline{g(x)}\, dx$. Now, let me define let us consider the linear operator as D that is differentiation operator on V. So, I have taken my linear operators derivative.

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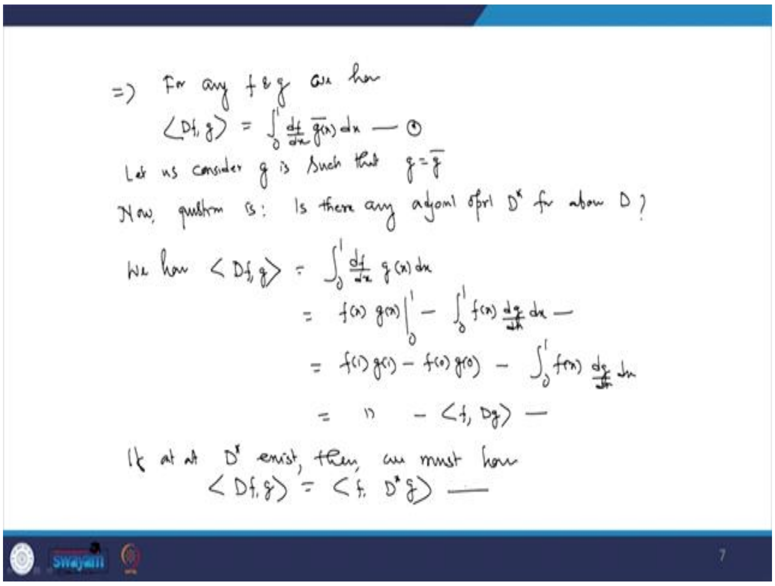

$\Longrightarrow$ For any f & g we have, < Df, g > = $\int_0^1 \frac{df}{dx}\overline{g(x)}\,dx$. Let us consider g is such that $g = \overline{g(x)}$, I mean it is the coefficient of the polynomial g is real. Now, question is, is there any adjoint operator D^* for above D? To answer these questions let us proceed what is definition of D^*? We have, < Df, g > = $\int_0^1 \frac{df}{dx} g(x)\,dx$ = $f(x)g(x)_0^1 - \int_0^1 f(x)\frac{dg}{dx}\,dx$ = $f(1)g(1) - f(0)g(0) - \int_0^1 f(x)\frac{dg}{dx}\,dx$ = $f(1)g(1) - f(0)g(0) -$ < f, Dg >

Now, if you want to say that D^* exists then we must have < Df, g > = < f, D^*g >, so, you must have in terms of the inner product of f and some operator operating on g. We must have this type of structures.

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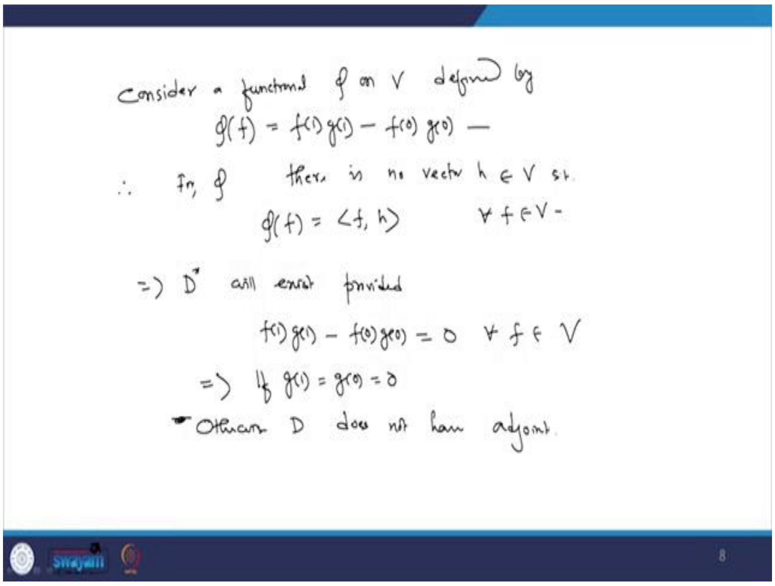

Now, consider if functional Ψ on V defined by Ψ(f) = $f(1)g(1) - f(0)g(0)$. So, this is a linear functionals on V. Now, this linear functional is similar what we have already done over the same space, if you recall why consider one linear functionals Ψ on that this type of space wher Ψ(f) = f(z_0) where z_0 is a fixed value and your field $\mathbb{C}$.

So, in that case we have seen that for that linear functional there is vector β such that the linear functional can be expressed as the inner product of that vector. So, therefore, for this linear function there is no fixed vector, say h such that this will be equal to inner product of f and h. So, this implies that Ψ, there is no vector say h ∈ V such that Ψ(f) = < f, h > ∀ f ∈ V.

$\Longrightarrow$ D^* will exist provided $f(1)g(1) - f(0)g(0)$ = 0 ∀ f ∈ V $\Longrightarrow$ g(1) = g(0) = 0. So, if we choose a fixed function g such that g(1) = g(0) = 0. So that is getting adjoint for the operator

D. Otherwise D does not have adjoint. So, now let me give some one interesting results about the adjoint operators in terms of the small theorems.

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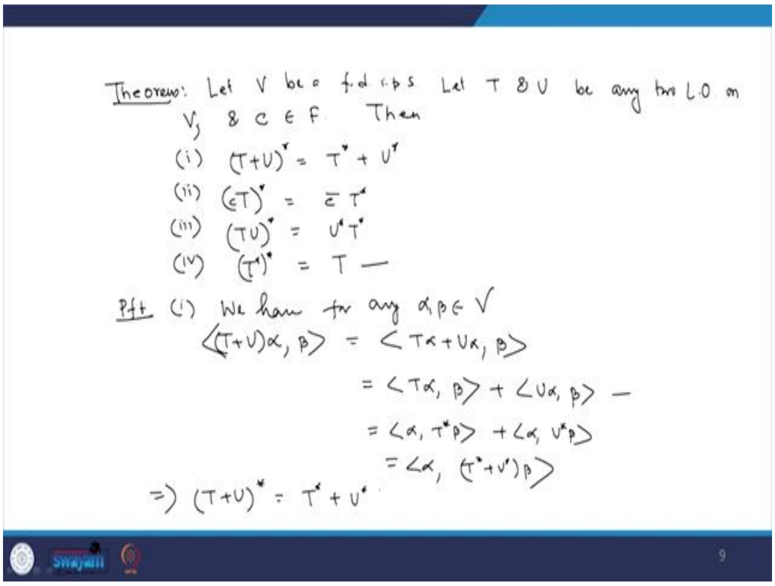

Let V be a finite dimensional inner product space. Let T & U be any two linear operators on V & c ∈ F on which the vector space defined. So, let V be the finite dimensional inner product space. Here I have taken the vector space is defined over the field F, it may be complex number, it may be real number. Let T and U be any two linear operator on V and c ∈ F.

Then, (i) $(T+U)^* = T^* + U^*$ (ii) $(cT)^* = \bar{c}\,T^*$(iii) $(TU)^* = U^*T^*$ (iv) $(T^*)^* = T$, so, this proof is very simple I will prove (i) & (ii) then we will leave it on you and you can take a resolution to cross-check the result.

We have for first we have $\alpha, \beta \in$ V, such that $<(\mathrm{T}+\mathrm{U})\alpha, \beta> = <\mathrm{T}\alpha + \mathrm{U}\alpha, \beta>$ $<\mathrm{T}\,\alpha, \beta>$ $+ <\mathrm{U}\alpha, \beta> = <\alpha, T^*\beta> + <\alpha, U^*\beta> = <\alpha, (T^* + U^*)\beta> \Longrightarrow (T+U)^* = (T^* + U^*)$, Exactly same way we can prove the all the rest of the results. So, we can complete it in your home.

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One nice property of this adjoint operator is like this. Consider T be a linear operator on a finite dimensional inner product space V. Let T^* be adjoint T. So, one can write down, T = $\frac{T+T^*}{2} + \frac{i}{2i}(T - T^*) = U_1 + iU_2$. Here, $U_1{}^* = \left(\frac{T+T^*}{2}\right)^* = \frac{T^*+T^{**}}{2} = \frac{T^*+T}{2} = \frac{T+T^*}{2} = U_1$

And similarly, $U_2{}^* = U_2$. An operator T on V is said to be self adjoint provided, $T = T^*$. So, from this above relations I mean T can be written as $U_1 + iU_2$. where U_1 & U_2 are self adjoint. But I am not saying that the operator T is a adjoint.

But if you give me any linear operator on a finite dimensional inner product space, I will be able to express as a sum of two self adjoint operators. Following the complex number pattern that $U_1 + iU_2$ where i is a basically $\sqrt{-1}$. One may also write the question. What is the beauty on it? Later and I will show you that if a operator is a self adjoint then the corresponding space will have an orthogonal order basis.

So, this orthogonal or orthonormal order basis will certainly simplify the structure of the operator to understand the operator very easily. This type of existence of orthonormal order basically useful. We will discuss inner future classes.