Advanced Linear Algebra
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Lecture – 50
Linear Functionals and Adjoints-III

Welcome to lecture series in Advance Linear Algebra. We have seen over a finite dimensional inner product space if we consider any linear functional. Then there is just a unique vector over the space so that linear functional is basically inner product of that fixed vector. And we also seen if a linear operator define over a finite dimensional inner product space. It has adjoint operator on it.

Today we will see how to find that unique vectors when a linear functional defined over a inner product space.

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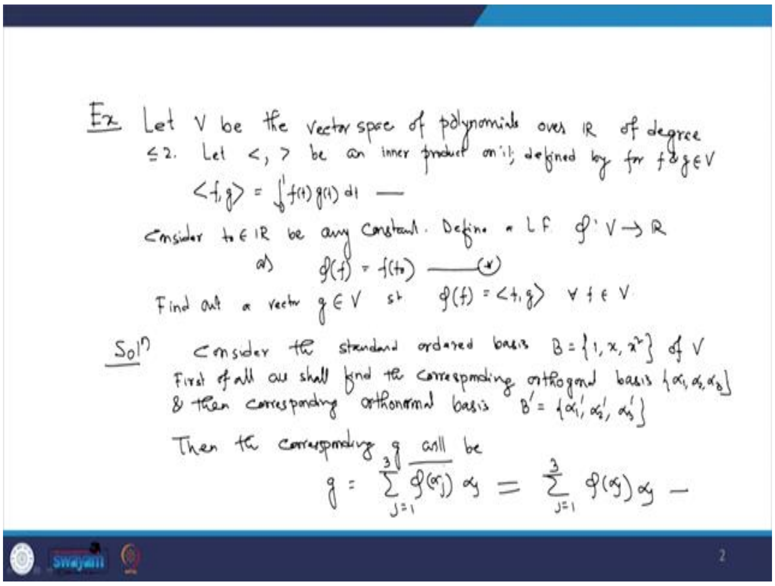

So, let me consider this problem, Ex:- Let V be the vector space of polynomial functions over the real line of degree ≤ 2 and the inner product defined in this space is defined by < f, g > = $\int_0^1 f(t)\, g(t)\, dt$ or a fixed real number see t_0 ∈ R which is constant if I consider that we define a linear functional Ψ: V → R as Ψ(f) =f(t_0) → (*)

So, since we already know this function Ψ: V → R is a linear functional. So now, we are interested to find a function or an element in V, g ∈ V s.t. Ψ (f) = < f, g > = ∀ f ∈ V. So now, if you recall the theorems, we need basically an orthonormal basis for the space V and then the unique element we recall that β here it will be something g.

This will be basically g = $\sum_{j=1}^{3} \overline{\Psi(\alpha_j)}\, \alpha_j$. So, how to proceed this one? Solution:- Consider the standard order basis, B = $\{1, \mathrm{x}, \mathrm{x}^2\}$ of V. First, we have to obtain from this order basis the corresponding orthogonal basis we shall find the corresponding orthogonal basis, say $\{\alpha_1, \alpha_2, \alpha_3\}$

And then corresponding orthonormal this is say B$'$ =$\{\alpha_1{}', \alpha_2{}', \alpha_3{}'\}$. So, first objective will be to how to find B$'$? To find the orthogonal basis we have to use the Gram Schmidt orthogonal procedure and then once that orthogonal vectors are obtained divide by the magnitude of the vectors will give you orthonormal basis.

So, suppose B$'$ is known to us then the corresponding g = $\sum_{j=1}^{3} \overline{\Psi(\alpha_j{}')}\, \alpha_j{}'$ = $\sum_{j=1}^{3} \Psi(\alpha_j{}')\, \alpha_j{}'$ since we have taken over the real.

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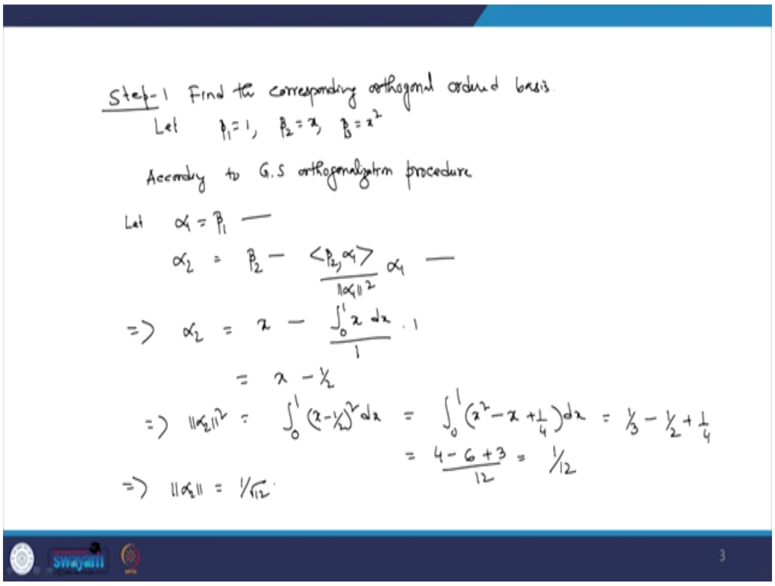

Step: -1 Find the corresponding orthogonal order basis. Let β_1= 1, β_2 =x, $\beta_3 = \mathrm{x}^2$. So, according to Gram Schmidt's orthogonal procedure. alpha 1 let me consider $\alpha_1 = \beta_1, \alpha_2 = \beta_2$ - $\frac{<\beta_2,\alpha_1>}{||\alpha_1||^2}\,\alpha_1 \implies \alpha_2 = x - \frac{\int_0^1 x\,dx}{1} = \mathrm{x} - \frac{1}{2}$

$||\alpha_2||^2 = \int_0^1 \left(\mathrm{x} - \frac{1}{2}\right)^2 dx = \int_0^1 (\mathrm{x}^2 - x + \frac{1}{4})dx = \frac{1}{3} - \frac{1}{2} + \frac{1}{4} = \frac{4-6+3}{12} = \frac{1}{12}$. $||\alpha_2|| = \sqrt{\frac{1}{12}}$

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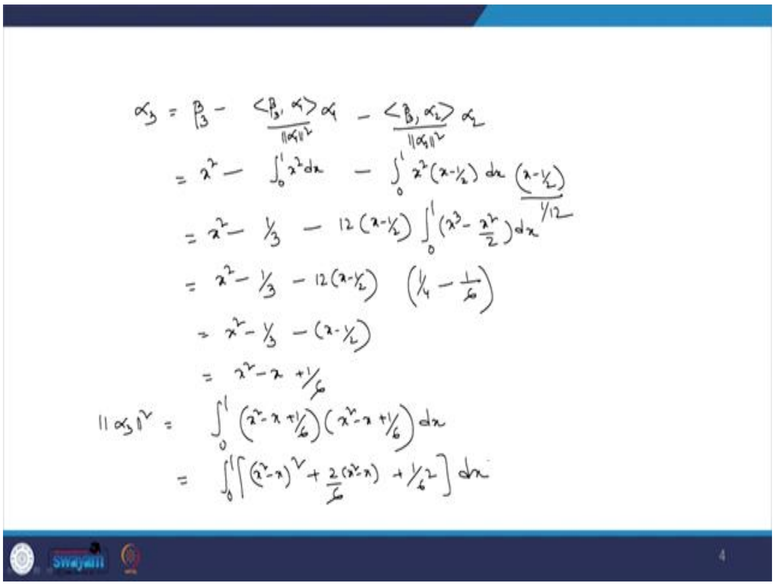

$\alpha_3 = \beta_3$ - $\frac{<\beta_3,\alpha_1>}{||\alpha_1||^2}\alpha_1$- $\frac{<\beta_3,\alpha_2>}{||\alpha_2||^2}\alpha_2$ = $x^2 - \int_0^1 x^2\,dx - \int_0^1 x^2(x-\frac{1}{2})dx\,\frac{(x-\frac{1}{2})}{\frac{1}{12}}$ = $x^2 - \frac{1}{3} - 12(x-\frac{1}{2})\int_0^1 (x^3 - \frac{x^2}{2})\,dx = x^2 - \frac{1}{3} - 12(x-\frac{1}{2})(\frac{1}{4}-\frac{1}{6}) = x^2 - \frac{1}{3} - (x-\frac{1}{2}) = (x^2 - x + \frac{1}{6})$

So, $\alpha_3 = (x^2 - x + \frac{1}{6})$ **(()) (11:48)** So, we have already obtained the corresponding orthogonal order basis $\alpha_1, \alpha_2, \alpha_3$. Now, also we have to calculate what is the $||\alpha_3||^2$? This is $||\alpha_3||^2 = \int_0^1 (x^2 - x + \frac{1}{6})(x^2 - x + \frac{1}{6})\,dx = \int_0^1\{(x^2 - x) + \frac{2}{6}(x^2 - x) + \frac{1}{(6)^2}\}\,dx$

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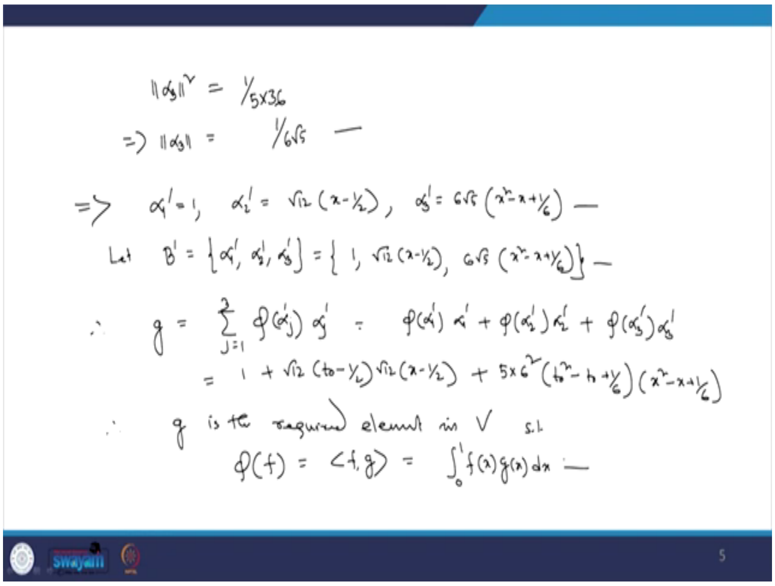

So, if, after simplifications, I will have in the end your $||\alpha_3||^{\,2} = \frac{1}{5x36} \Longrightarrow ||\alpha_3|| = \frac{1}{6\sqrt{5}}$. So, we know so then we can immediately calculate your $\alpha_1{}' = 1, \alpha_2{}' = \sqrt{12}\left(x - \frac{1}{2}\right), \alpha_3{}' = 6\sqrt{5}\,(x^2 - x + \frac{1}{6})$

You can immediately cross check whether this $\alpha_1, \alpha_2, \alpha_3$ are orthogonal or not that we can check by taking the $< \alpha_1{}', \alpha_2{}' > = 0$ or $< \alpha_2{}', \alpha_3{}' > = 0$, see they are orthogonal. And once they are orthogonal and then you also can calculate the $||\alpha_1||$, $||\alpha_2||$, $||\alpha_3||$ and if you divide by the corresponding vectors.

I will have $\alpha_1{}', \alpha_2{}', \alpha_3{}'$. So, I have now so, let $B' = \{\alpha_1{}', \alpha_2{}', \alpha_3{}'\} = \{B' = \alpha_1{}' = 1, \alpha_2{}' = \sqrt{12}\left(x - \frac{1}{2}\right), \alpha_3{}' = 6\sqrt{5}\,(x^2 - x + \frac{1}{6})$. So then, according to that last results, so, your g will be that unique, fixed vector in V, g will be what so, $g = \sum_{j=1}^{3} \Psi(\alpha_j{}')\,\alpha_j{}' = \Psi(\alpha_1{}')\,\alpha_1{}' + \Psi(\alpha_2{}')\,\alpha_2{}' + \Psi(\alpha_3{}')\,\alpha_3{}' = 1 + \sqrt{12}\left(t_0 - \frac{1}{2}\right)\sqrt{12}\left(x - \frac{1}{2}\right) + 5x6^2\,({t_0}^2 - t_0 + \frac{1}{6})(x^2 - x + \frac{1}{6})$ since it is at the value of t_0.

g is the required element in V such that your $\Psi(f) = < f, g > = \int_0^1 f(x)g(x)\,dx$. I hope this example will certainly give a clear picture about this concept. We have also seen that over a finite dimensional inner product space, if it operator T is defined then it is adjoint exists.

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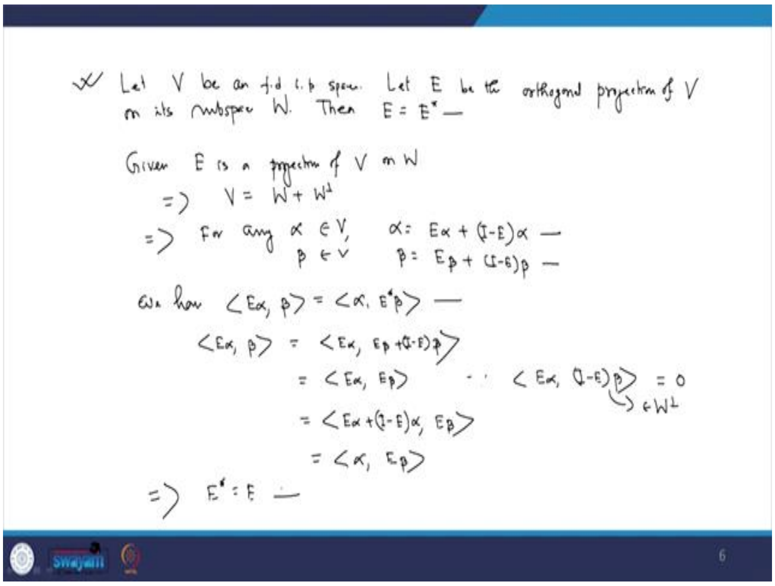

And also we have seen that if you consider an order basis which is consisting of all orthonormal vectors then what is the correlation between the matrix representation of T and T^* with respect to orthogonal exists. Now, let us see for a specific operators let me consider let V be a finite, dimensional inner product space. Let E be the orthogonal projection of V on it is subspace W. Then $E = E^*$

I mean the corresponding adjacent operator of the E is basically E. Let us see how it is given is a projection of V orthogonal projections on W $\Longrightarrow$ I can write down V = $W \oplus W^{\perp}$ $\Longrightarrow$ For any $\alpha \in$ V, $\alpha = E\alpha + (I - E)\alpha$

We know what is the meaning of W and $W^{\perp}$ any element of W will be orthogonal to it is orthogonal complement And similarly $\beta \in$ V, $\beta = E\beta + (I - E)\beta$. We have $< E\alpha, \beta > = <\alpha, E^*\beta >$, we want to know what is E^* for this problem. So, $< E\alpha, \beta > = < E\alpha, E\beta + (I - E)\beta > = < E\alpha, E\beta > = < E\alpha + (I - E)\alpha, E\beta > = < \alpha, E\beta >$

since $< E\alpha, (I - E)\beta = 0$ because this $(I - E) \in W^{\perp}$ $\Longrightarrow E = E^*$

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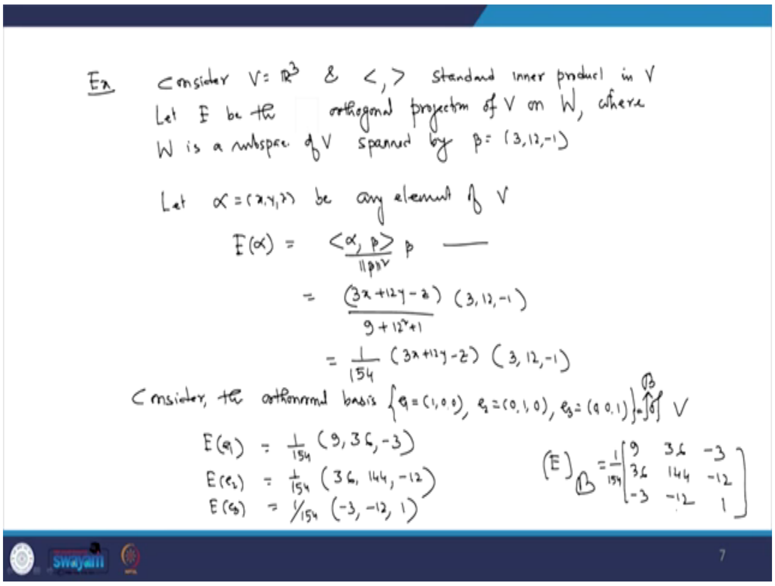

So, if I consider, if orthogonal projection operator of a vector space V on a it is subspace W then the adjoint of the projection operator E will be equal to that operator itself. So, let me take the Ex:- Consider V = R^3 & < , > standard inner product standard inner product in V. Let E be the orthogonal projection of V on W, where W is a subspace of V spanned by β = (3, 12, –1).

Let us see what is the matrix representation of this operator with respect to standard order basis of V. Of course, I will take standard orthonormal order basis of V and you also see what is the corresponding matrix representation of E^*? According to theorems, we have seen that $E = E^*$. Now, let us check whether this is correct through in this example or not. So, let α = (x, y. z) be any element of V, since orthogonal projection of α.

I am talking about the best approximations of α in W. So, then $E\alpha = \frac{<\alpha,\beta>}{||\beta||^2}\beta = \frac{3x+12y-z}{9+12^2+1}(3, 12, -1) = \frac{3x+12y-z}{154}(3, 12, -1)$. So, consider the orthonormal basis$\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ =B of V.

So, let me consider this standard orthonormal order basis like this, so that $E(e_1) = \frac{(9,36,-3)}{154}$, $E(e_1) = \frac{(36,144,-12)}{154}$. $E(e_1) = \frac{(-3,-12,1)}{154}$. So, if I consider the matrix representation of E with respect to the order basis B.

i.e. $[E]_B = \frac{1}{154}\begin{bmatrix} 9 & 36 & -3 \\ 36 & 144 & -12 \\ -3 & -12 & 1 \end{bmatrix}$. So, I see this is a matrix which is symmetric so, definitely and over the real, so, it will be equal to the transpose I mean.

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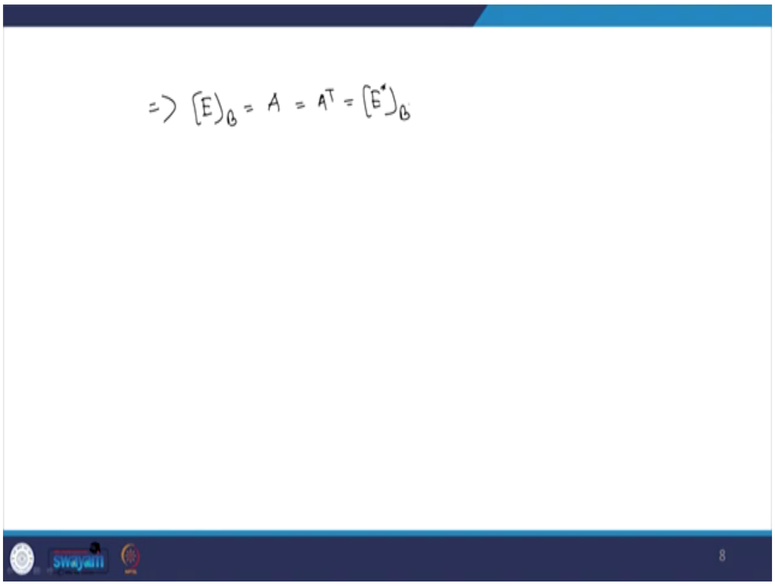

$\Longrightarrow [E]_B = A = A^T = [E^*]_B$. So, we see that here the matrix representation of the projection operator E is basically symmetric matrix. So, since it is defined over the real line and therefore, matrix representation of E and E^T both are same.

But here, if I do not take orthonormal basis then we it is not easy to correlate. The matrix representation of E and it is corresponding E^*. Since we do not have much time here so, I could not write down it here but certainly you will get it in the assignment set.