

**Advanced Linear Algebra**  
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**Lecture – 5**  
**Vector Spaces - 2**

So, welcome to our lecture series. So, we are discussing about the vector space. We have defined what is vector space, we have taken a couple of examples of it. Before taking more examples and to give some sort of geometrical representation wherever it is possible, let me define one more terminology called subspace.

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Subspace: Let  $V$  be a vector space over a field  $F$ . A nonempty subset  $W$  of a vector space  $V$  is said to be subspace of  $V$  provided  $W$  satisfies all conditions or axioms of vector space w.r.t. given vector addition & scalar multiplication.

Ex Let  $V = F^2$ , with vector addition & scalar multiplication on  
 for  $u, v \in V$ ,  $u + v = (u_1 + v_1, u_2 + v_2)$  — (i) Vector addition  
 $cu = (cu_1, cu_2)$  — (ii) scalar multiplication  
 $u = (u_1, u_2)$   
 $v = (v_1, v_2)$

Let  $W = \{(x, 0), x \in F\}$

Criteria for a subset of vector space to be a subspace of it:

Let  $V$  be a vector space over a field  $F$ . If I say  $V$  be a vector space, then certainly there are two binary operations also there that is one vector addition and scalar multiplication. Then a nonempty subset  $W$  of a vector space is said to be subspace of  $V$  provided  $W$  satisfies all criterion or axioms of vector space with respect to vector addition and scalar multiplication.

So, what I am saying that a nonempty subset  $W$  of a vector space  $V$  over a field of  $F$  is said to be a subspace of  $V$  provided that  $W$  itself satisfy all the criteria of the vector space, even  $W$  itself is again vector space. So, let me consider that  $V = F^2$ , the space of all two tuples over  $F$  and vector addition and scalar multiplication are defined like what we did last time with vector addition and

scalar multiplication.

As for  $u$  and  $v \in V$ ,  $u + v = (u_1 + v_1, u_2 + v_2)$  this is the vector addition and any  $c \in F$ ,  $c * u = (cu_1, cu_2)$  where  $u = (u_1, u_2)$  and  $v = (v_1, v_2)$ . So, this is my scalar multiplication. We know with respect to these two binary operations  $V$  is a vector space. So, if I just consider geometrically say my  $V$ —if  $F$  is equal to a real number, then this I can say as simply  $x$ - $y$  plane type of things.

So, it is basically some sort of a  $x$ - $y$  plane type, when I say all points in the  $x$ - $y$  plane that is my  $V$ . Now, if I consider all the points on the  $x$ -axis, let  $W = \{(x, 0), x \in F\}$ . I mean all the tuples where the second entry is 0, this means that this is all the points in the vector space. So, we can immediately check it satisfies all the axioms of the definition of the vector space. Therefore,  $W$  will be also a subspace of the vector space  $V$ .

Of course, it is very tedious to check all these 10 axioms for the set  $W$  suppose, now question is it possible to give a simple formula, one can use that formula to check where a given subset of a vector space is a subspace of the vector space or not. So, let me quickly give that formula first so that after that if I take a different example, it will be easy to understand whether a given subset of a vector space is subspace or not. Criteria for a subset of a vector space to be a subspace of it, so what is the criteria for this one?

If I consider  $W$  is a subset of a vector space  $V$ , then certainly with respect to scalar multiplications that  $W$  will satisfy, apart from closure property all other properties will hold good. All are axioms of one vector space only. So, if I can somehow show that a subset is closed and is an abelian group space with respect to both vector addition as well as scalar multiplication and with respect to vector addition the abelian group, then we can say that that subset is a subspace of the corresponding given vector space.

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Theorem: A nonempty subset  $W$  of a vector space  $V$  over  $F$ , is a subspace of  $V$  if and only if for any  $u, v \in W$  &  $c \in F$

$$cu + v \in W \quad (*)$$

pf: If the criteria  $cu + v \in W$  is satisfied we want to show  $W$  is a vector subspace of  $V$ .

Given  $W$  is nonempty, so let  $v \in W$ , & we know  $-1 \in F$

According to  $(*)$   $(-1)v + v \in W$

$\Rightarrow -v + v = 0 \in W$ , Here "0" is zero element of  $V$

$\Rightarrow$  for any  $c \in F$  &  $u \in V$

$cu + 0 = cu \in W$ ,  $\Rightarrow$   $W$  is closed w.r.t scalar multiplication

for any  $u, v \in W$ ,  $u + v \in W$ ,  $\Rightarrow u + v \in W$

$\Rightarrow$   $W$  is closed w.r.t vector addition.

So, I can consider one standard result like this in terms of theorems like this. A nonempty subset  $W$  of a vector space  $V$  over  $F$ , is a subspace of  $V$  if and only if for any  $u$  &  $v \in W$  and  $c \in F$ ,  $c * u + v \in W$ . If this criteria is satisfied, then immediately I can say that the subset  $W$  of the given vector space will be subspace of the vector space  $V$  or since I am talking about if and only if, if  $W$  is a subspace then also this criteria that  $c * u + v \in W$ .

Now, let me give if the criteria where  $c * u + v \in W$  is satisfied we want to show  $W$  is a vector space, as it is  $W$  is a subspace of  $V$ . I mean to say I have to check with this given criteria that  $W$  satisfied closure property with respect to multiplications and  $W$  is an abelian group because rest of the axioms with respect to scalar multiplication is basically axioms of the  $V$  and  $W$  is part of it, it will be satisfied.

So, therefore I have to basically check only this set of properties. It is given to me for any  $c \in F$  and  $u$  &  $v \in W$ ,  $c * u + v \in W$ . Given  $W$  is nonempty, so let  $v \in W$  and we know  $-1 \in F$ . So according to this criterion, according to star I can say your  $(-1) * v + v \in W$ . So, one this is star. please sorry So, this implies that see  $-1 * v = -v$ , so this is equal to  $-v + v = 0 \in W$ .

Here  $0$  is zero element of  $V$ . So the zero element of  $V$  also zero element of  $W$  and it is also inside

W. So, this implies for any  $c \in F$  and  $u \in V$ ,  $c * u + 0 = c * u \in W$ , implies W is closed with respect to scalar multiplication. For any  $u$  &  $v \in W$  one can write down  $c = 1$  and  $1 * u + v \in W$ . This implies  $(u + v) \in W$ . This implies W is closed with respect to vector addition.

So, W is closed with respect to scalar multiplication as well as vector addition. Now I want to show; see associative property will hold good by default because this associative property is inherent property of the vector space V, so I do not have to prove whether W satisfied associative property or not. So that property holds good by default.

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Associative prop holds good in W also,  $\because W \subset V$ .  
 For additive inverse, we have for  $u \in W$   
 $(-1)u + 0 \in W$   
 $\Rightarrow (-1)u \in W$  i.e.  $-u \in W$   
 $\Rightarrow$  additive inverse also exist  
 $\because W \subset V$ , which satisfies commutative property so commutative prop will also hold good for W.  
 $\therefore$  W is an abelian group w.r.t. vector add<sup>n</sup> & is closed w.r.t. scalar multiplication. Thus W is a subspace of V.  
 $\Leftarrow$ : Given W is a subspace of V. So it is closed w.r.t. vector add<sup>n</sup> & scalar multiplication. Therefore for any  $u, v \in W$  &  $c \in F$   
 $c(u+v) \in W$   
 $c u + v \in W$

Associative property holds good in W also, since W is a subset of V. Now, I have to check existence of additive inverse. For additive inverse we have for  $u \in W$ ,  $(-1) * u + 0 \in W$ . So, this implies  $(-1) * u \in W$ , i.e.  $-u \in W$ . So, this implies additive inverse also exist. Since W is a subset of V, which satisfied commutative property, so commutative property will also hold good for W.

So, W is an abelian group with respect to vector addition and is closed with respect to scalar multiplication. Thus W is a subspace. So, W is a subspace of V. Now, other way it is given W is a subspace of the V, so it is closed with respect to vector addition and scalar multiplication. Therefore for any  $u$  &  $v \in W$  and  $c \in F$ ,  $c * u + v \in W$ . So, this criterion holds good in this case.

So, we see that if nonempty subset of a vector space will be a subspace provided for any  $u$  &  $v$

$\in W$  and  $c \in F$ ,  $c * u + v \in W$ . For sake of probity or simplicity I will not write down star, I will simply write  $c u$ . So, if I write  $cu + v$  means there is a binary operation star in a scalar multiplication in between  $c$  and  $u$ . So, now onwards I will lose this is basically  $cu + v \in W$ . So, this is with due permission from you I am typing this one.

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Ex Let  $V$  be a vector space over  $F$ .  
 Let  $W = \{0\}$  is collection of zero element of  $V$ .  
 claim  $W$  is a subspace of  $V$ .

It is called as zero subspace

Similarly,  $W = V$ , it is also subspace of  $V$ .

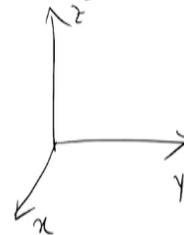
Ex Let  $V = F^3$ , Let  $W = \{v \in V : v = (0, x, y), x \in F, y \in F\}$

We know for any  $u, v \in W$

$$u = (0, u_2, u_3)$$

$$v = (0, v_2, v_3)$$

$$u + v = (0, u_2 + v_2, u_3 + v_3) \in W$$



So, let me take different example. Let  $V$  be a vector space over  $F$ , over  $F$  means  $F$  is the field of scalar and vector space means there is definitely two binary operation there vector addition and scalar multiplication and satisfy all the criteria of the vector space. Let  $W = \{0\}$ , that is collection of zero element of  $V$ . Claim  $W$  is a subspace of  $V$ . How to prove it? I have to check whether the criteria for any  $u$  &  $v \in W$  and  $c \in F$ ,  $c * u + v$  belongs to  $W$  or not.

Since  $W$  control is single element zero, zero element basically of the vector space  $V$ , so any scalar  $c$ ,  $c * 0 + 0 = c * 0$  which is equal to again zero element. So, therefore this  $W$  is a subspace of the vector space  $V$ . So, this is called the trivial subspace. It is called as zero subspace. Similarly, if I take  $W = V$  then also it is a subset of  $V$  and it is also subspace of  $V$ . So, these are basically trivial subspaces of the corresponding vector space  $V$ .

Now let me take other example. Let  $V$  equal to collection of all three tuples, so this is clearly say  $F^3$ , it may be real number or a complex number. Now, let  $W = \{v \in V, : v = (0, x, y), x \in F, y \in F\}$ . Now, we can quickly check whether this  $W$  is a subspace or not. We have for any  $u$  &  $v \in W$ ,  $u = (0, u_1, u_2)$  and  $v = (0, v_1, v_2)$ .

So, the binary operation over n-tuple space that is adding component wise, so if I do that one is my binary operation vector addition here, then we have  $u + v = (0, u_2 + v_2, u_3 + v_3) \in W$ . So, what I mean to say you if I consider over the Euclidean space  $x, y, z$  all the points in Euclidean space is my given vector space say  $V = F^3$ , here of I consider  $F$  equal to real number.

Now, if consider the  $W$ ,  $W$  is set of all the points which have first entry 0, this means that it will be over the; so it has all the points in the  $y$ - $z$  plane. So, you see that all the points in  $y$ - $z$  plane is subspace of the vector space  $V$ . Exactly same way one can check all the points on the  $x$ - $z$  plane is also subspace of  $V$  and all the points over the  $x$ -axis is also subspace of  $F^3$ , all the points over the  $y$ -axis is also subspace of  $F^3$ ,. So, we can have many subspaces for this.

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Ex Let,  $V = F^n$ . Let  $W$  be the sol<sup>n</sup> of the system of eqns  
 $AX = 0$ , where  $A$  is  $m \times n$  matrix over  $F$  &  $X$  is  $n \times 1$  column matrix  
 claim  $W$  is a subspace of  $V$   
 For any  $X_1, X_2$ , which are sol<sup>n</sup>s of  $AX = 0$   
 we have  $AX_1 = 0$  &  $AX_2 = 0$   
 For any  $c \in F$   
 $cX_1 + X_2 = cX_1 + X_2$   
 $A(cX_1 + X_2) = A(cX_1) + AX_2$   
 $= cAX_1 + AX_2 = c \cdot 0 + 0 = 0$   
 $\Rightarrow Y = cX_1 + X_2$  is also sol<sup>n</sup> of  $AX = 0$   
 $\Rightarrow W$  is a subspace of  $V$ .

Let  $V$  equal to  $F^n$ , I mean set of space are all n-tuples integrals over  $F$ . So, this is a vector space with respect to vector addition and multiplication what I have defined last time. I would say that is my standard addition over the  $F^n$ , and that is multiplication that I consider as a standard scalar multiplication for the space of all n-tuple. Let  $W$  be the solution of the system of equation  $AX=0$  where  $A$  is  $m \times n$  matrix over  $F$  and  $X$  is basically  $n \times 1$  column matrix.

Now, this  $W$  what I am saying is that the solutions of this system  $AX=0$  will be subset of certainly the vector space  $V = F^n$ , My claim is that it is also a subspace. Claim,  $W$  is a subspace

of  $V$ . For any  $X_1$  and  $X_2$  which are solutions of  $AX=0$  we have  $AX_1=0$  and  $AX_2=0$ . For any  $c \in F$ , we have  $c * X_1 + X_2 = c X_1 + X_2$ . My claim is that this is also solution of the system  $AX=0$ .

I have to prove that  $A(cX_1 + X_2) = A(cX_1) + AX_2$ . So, this is equal to again  $c$  I can take common, so  $c(AX_1) + AX_2 = c * 0 + 0 = 0$ . So, this implies your  $Y = c X_1 + X_2$ . is also solution of  $AX=0$ . This implies that  $W$  is a subspace of vector space this one.

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Ex Let us represent set of all  $m \times n$  matrices over  $F$  by  $F^{m \times n}$   
 Let  $V = F^{m \times n}$ , define vector add<sup>n</sup> & scalar multiplications  
 below  
 For  $A, B \in V$ , where  $A = (a_{ij})$   $1 \leq i \leq m, 1 \leq j \leq n$   
 $A + B = (a_{ij} + b_{ij})$  — (i)  
 $c \in F, B \in V$   
 $c * A = c * (a_{ij}) = (ca_{ij})$  — (ii)  
 Show that  $V$  is a vector space (H.W)  
 Consider  $V = F^{n \times n}$ , which also a vector space  
 Consider  $W$  as collection of all symmetric matrices  
 i.e.  $W = \{ A \in V : a_{ij} = a_{ji} \quad \begin{matrix} 1 \leq i \leq n \\ 1 \leq j \leq n \end{matrix} \}$   
 Claim  $W$  is a subspace of  $V$ . (H.W)

Let me take say another example. Let us represent set of all  $m \times n$  matrices over  $F$  by  $F^{m \times n}$  notation. Let  $V = F^{m \times n}$ . Define vector addition and scalar multiplication as below. For  $A$  &  $B \in V$ , where say  $A = (a_{ij})$ ,  $1 \leq i \leq m, 1 \leq j \leq n$ . So, defining  $A + B = (a_{ij} + b_{ij})$ , so this is my addition.

And scalar multiplication as for  $c \in F$ , &  $A \in V$ ,  $c * A = c * (a_{ij}) = c (a_{ij})$ . So,  $c$  is multiplied to each entry of the matrix, of course I will say that this is the standard scalar multiplication for this set of elements. Now, with respect to these two binary operations, one can easily show that  $V$  is a vector space, so it is done basically, I am giving as a homework because the proof is exactly the same what we did for the  $F^n$ .

So, now I have taken my vector spaces  $F^{m \times n}$  over the  $F$  and the vector addition and multiplication like this thing. Consider  $V = F^{n \times n}$ , so I have taken  $m = n$  which is also a vector space. Then consider  $W$  as collection of all symmetric matrices that is  $W = \{ A \in V, a_{ij} = a_{ji} \quad 1 \leq i \leq n, 1 \leq j \leq n \}$   
 Claim,  $W$  is a subspace of  $V$ . Again one can show that  $c$  times  $A_1 + A_2$  where  $A_1, A_2$  belongs to  $W$ , I mean  $A_1$  is symmetric,  $A_2$  also symmetric, then  $c A_1 + A_2$  is also a symmetric matrix. So, I can

leave this as also a homework.

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Linear Combination: A vector  $\beta \in V$ , where  $V$  is a vector space over  $F$ , is said to be linear combination of vectors in  $V$ , provided there exist some distinct vectors say  $v_1, v_2, \dots, v_n$ , & scalars  $c_1, c_2, \dots, c_n \in F$ , such that

$$\beta = c_1 v_1 + c_2 v_2 + \dots + c_n v_n$$

Subspace spanned by a subset of a vector space: Let  $V$  be a vector space over  $F$ . Let  $S$  is a nonempty subset of  $V$ . Let  $L$  denotes the collection of all possible linear combinations of vectors of  $S$  i.e.  $L = \{ c_1 v_1 + c_2 v_2 + \dots + c_n v_n, \text{ where } v_1, v_2, \dots, v_n \in S \text{ and } c_i \in F \}$

This collection is a subspace of  $V$ .

Let  $u, v \in L$   
 $u = \sum_{i=1}^n x_i v_i, v_i \in S, i=1, \dots, n, v = \sum_{i=1}^n y_i v_i$

Let me use a concept called linear combination. A vector  $\beta \in V$  where  $V$  is a vector space over  $F$  is said to be linear combination of vectors, there exist some distinct vectors say

$(v_1, v_2, \dots, v_n), (c_1, c_2, \dots, c_n) \in F$  such that  $\beta = (c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$ . Basically it is  $(c_1 * v_1 + c_2 * v_2 + \dots + c_n * v_n)$ . where the star is basically scalar multiplication. For sake of simplicity, I have removed the star. If I write simply  $(c_1 v_1 + c_2 v_2 + \dots + c_n v_n)$ . So this is the definition of when one can say a vector of a vector space a linear combinations of some vectors.

So fine, let me introduce some more and more terminology subspace spanned by a subset of a vector space. What is the meaning of this one? Let  $V$  be a vector space over field  $F$ . Let  $S$  is a subset of  $V$ .

I am not saying that capital  $S$  is the subspace of  $V$ , I am simply saying that it is subset of  $V$ . Let capital  $L$  denote the collection of all possible linear combination of vectors of capital  $S$ . So, if I say that it is a possible collection of all linear combination of vectors of  $S$  means that is over this field is  $F$  that is I can write  $L = \{ c_1 v_1 + c_2 v_2 + \dots + c_n v_n, \text{ where } (v_1, v_2, \dots, v_n) \in S \text{ and } c_i \in F \}$ .

So, this is a collection of all the linear combinations of the vectors of  $S$ . This collection is a subspace of vectors of  $V$ , how? Consider  $S$  is a nonempty subset of  $V$ . Now, if I consider  $S$  as nonempty subset of  $V$ , so for any linear combinations, I will say that since the vector space is closed with respect to vector addition and scalar multiplication, so each linear combination is also element of  $V$ . So, therefore I can say  $L$  is nonempty.

Let  $u$  &  $v \in L$ . So  $u = \sum_{i=1}^n x_i v_i$ , such that  $v_i \in W$ ,  $i = 1$  to  $n$ . So, this  $n$  is fixed to be some number. It  $S$  may contain 1, 2, 3 infinite whatever it maybe. So, I am saying that your  $u$  is basically linear combination of some element of capital  $S$ . So, I can write  $u$  is equal to something like this thing and  $v$  equal to also we will have some sort of similar expression. So, I will write down  $v = \sum_{i=1}^n x_i v_i$ .

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$$u+v = \sum_{i=1}^n x_i v_i + \sum_{i=1}^n y_i v_i = \sum_{i=1}^n (x_i + y_i) v_i \in L$$

$\Rightarrow$  for any  $c \in F$

$$c \cdot u = c \sum_{i=1}^n x_i v_i = \sum_{i=1}^n c x_i v_i \in L$$

$\Rightarrow L$  is a subspace of  $V$ .

$\checkmark$  Let  $V$  be a vector space over  $F$  let  $W_1, W_2$  be any two subspaces of  $V$ . Then  $W_1 \cap W_2$  is also a subspace of  $V$ .

pf! for any  $u, v \in W_1 \cap W_2$  &  $c \in F$

$$c \cdot u + v \in W_1$$

$$c \cdot u + v \in W_2$$

$\Rightarrow c \cdot u + v \in W_1 \cap W_2$

Then  $u + v = \sum_{i=1}^n x_i v_i + \sum_{i=1}^n y_i v_i = \sum_{i=1}^n (x_i + y_i) v_i \in L$ . And for any  $c \in F$ ,  $c \cdot u = c \sum_{i=1}^n x_i v_i = \sum_{i=1}^n c x_i v_i \in L$ . This is again a linear combination where this is now new constant, new coefficient  $c x_i$ . So it is also linear combination the vectors of  $W$ , so this also belongs to  $L$ . Thus this implies  $L$  is a subspace of  $V$ . So, collection of all possible linear combinations of the vectors of  $W$  is a subspace of the corresponding vector space. Now, an interesting question may arise is like this. Let  $V$  be a vector space  $F$  and let  $W_1, W_2$  be any two subspaces of  $V$ .

Then  $W_1 \cap W_2$  is also a subspace of  $V$ . How to prove it? Proof is very straightforward. Since  $W_1$  and  $W_2$  both of them are basically subspaces of vector space  $V$ , for any  $u$  &  $v \in W_1 \cap W_2$  &  $c \in F$ ,  $c \cdot u + v \in W_1$  and  $c \cdot u + v \in W_2$ . So, this implies  $c \cdot u + v \in W_1 \cap W_2$ . So, we see that intersection of two subspaces is again a subspace.

In fact, if I take the intersection of the  $n$  subspaces again it will be a subspace. Now, what can you say if I say that union of two subspaces, will it be a subspace? Now, this question I can answer through geometry.

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What about union of two subspaces of a vector space?

$$\text{Let } V = F^3$$

$$W_1 = \{u \in V, \text{ s.t. } u = (x, 0, 0)\}$$

$$W_2 = \{v \in V, \text{ s.t. } v = (0, y, 0)\}$$

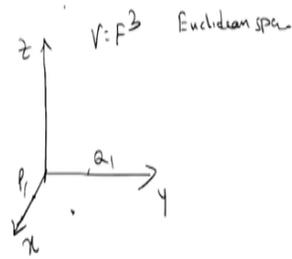
Let  $W = W_1 \cup W_2$ , will it be a subspace

Ans No.

$$\text{Let } u \in W_1 \text{ \& } v \in W_2$$

$$\Rightarrow u = (x, 0, 0), v = (0, y, 0)$$

$$u + v = (x, y, 0)$$



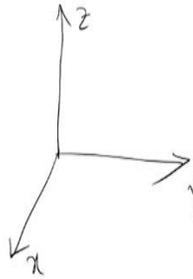
What about union of two subspaces of a vector space? Say let me take our very well-known vector space that is  $F^3$  or  $R^3$  I can say. So,  $V = F^n$ , set of all three tuples, so this is basically the collection of all the points in Euclidean space, this is the Euclidean space. Now, if I consider so let my  $V = F^3$ , I have taken here and  $W_1 = \{u \in V, \text{ s.t. } v = (x, 0, 0)\}$  and  $W_2 = \{v \in V, \text{ s.t. } v = (0, x, 0)\}$ . Something like this.

So, this means that  $W_1$  is my x axis. So, this is my x-axis, this is y axis, this is z-axis. So, you see that  $W_1$  is the x-axis and  $W_2$  is my y-axis. Now, if I consider let  $W = W_1$  union  $W_2$ , then will it be a subspace? Answer is no, how? Let  $u \in W_1$  and  $v \in W_2$ . So, this implies that  $u = (x, 0, 0)$  and  $v = (0, y, 0)$ . So,  $u + v = (x, y, 0)$ .

So, this is a point in the x-y plane. So, this point belongs to somewhere here. So, this is a point, this point u, so let me present u by the point say  $p_2$  and this is a point say  $q_1$  that is my v. Then the sum of two points is a point here which does not belong to  $W_1 \cup W_2$ . So, therefore union not necessarily to be a subspace, but it will be subspace provided I can convey like this you know. Let me just take an example and then show the what will be the possible answer.

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Ex Let  $V = F^3$   
 $W_1 = \{(x, y, 0)\}$   
 $W_2 = \{(x, 0, 0)\}$



W Sum of Subset of vector space:

Let  $V$  be a vector space over  $F$ .

Let  $W_1, W_2, \dots, W_k$  be subset of  $V$ .

Then, the collection of all pts  $w$  s.t.

$$w = w_1 + w_2 + \dots + w_k, \quad w_i \in W_i$$

is called an ~~non~~ sum of subsets of  $V$ .

If the subsets are subspaces, then it is called sum of subspaces.

Let  $V = F^3$ , and  $W_1 = (x, y, 0)$  and  $W_2 = (x, 0, 0)$ . Now see if I take  $W_1$  union  $W_2$ , so  $W_1$  is basically here x-y plane whereas  $W_2$  is my x-axis. So, union will be again in the x-y plane. So, it will be a subspace again. So, if I consider two subspaces  $W_1, W_2$  the union will be subspace provided one of them will contain the other that is the basic criteria.

Let me introduce another terminology called sum of subset of a vector space. Let  $V$  be a vector space over  $F$ . Let  $W_1, W_2, \dots, W_k$  be subset of  $V$ . Then the collection of all points such that your  $W = w_1 + w_2 + \dots + w_k$ ,  $w_i$  belongs to  $W_i$ ,  $i$  is the collection of all the point,  $W$  is called as sum of subsets of  $V$ . If the subsets are subspace, then it is called sum of subspaces.

So, we have defined what is subspace, we have defined what is intersection of subspaces, results of subspace. We also seen that union of two subspaces will be subspace provided one is inside of other. We have also defined what is sum of subset and sum of subspaces. So, we will continue in the next class as of today. Thank you.