

Advanced Linear Algebra
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Lecture – 49
Linear Functionals and Adjoints - II

Welcome to lecture series of Advanced Linear Algebra. In my last lecture, we have discussed about the correlation between any finite dimensional inner product space and a linear functional on that space. We have seen for a given linear functional on the finite advanced linear product space. There is a unit vector in the space and your linear functional is expressed as inner product with the fixed vector.

Then we raise the question whether this finite dimension inner product space is replaced by infinite dimensional inner product space. Then, is it true that any linear functional on that inner product space will be in the space of some inner product with some fixed vector in the space? I told you that answer is no. I am going to say it is not true always.

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Ex Let V be the space of all polynomial fns over the field of Complex number. Let $\langle \cdot, \cdot \rangle$ be inner product defn by $\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt$ —

Consider $z \in$ space of Complex number (F)

Defn a fn $L: V \rightarrow F$
 $L(f) = f(z)$ — (1)

So, L is a linear functional

Claim for this L , \exists no $g \in V$ st
 $L(f) = f(z) = \langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt$ —

If such g exist, then for $h = (z-2)$

So, to support my answer let me consider, for example, let V be all polynomial functions over the field of complex number. And let \langle, \rangle defined by $\langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt$. Consider z belongs to the space of complex number(F), I mean field next number is a fixed complex number I have consider.

Define a function $L: V \rightarrow F$ defined by your $L(f) = f(z)$ for z is a basically fixed point in the complex number which is some constant quantity. So, certainly we have already seen L defined a linear functionals. So, L is a linear functionals already we have seen. Now, claim for this L ,

\exists no $0 \neq g \in V$ such that $L(f) = f(z) = \langle f, g \rangle = \int_0^1 f(t) \overline{g(t)} dt$. If such g exists, then for $h = (x - z)$.

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$$\text{We have } (hf)(z) = 0 \quad \because h(x) = (x-z)$$

$$\Rightarrow L(hf) = (hf)(z) = 0 = \langle hf, g \rangle =$$

$$\Rightarrow 0 = \int_0^1 h(x)f(x) \overline{g(x)} dx \quad \forall f \in V$$

$$\Rightarrow \text{For } f = \overline{h}g, \text{ then}$$

$$0 = \int_0^1 h(x) \overline{h(x)} g(x) \overline{g(x)} dx$$

$$\Rightarrow 0 = \int_0^1 |h(x)|^2 |g(x)|^2 dx$$

$$\Rightarrow |h(x)||g(x)| = 0 \Rightarrow |g(x)| = 0 \quad \forall x \in (0,1)$$

We have, $(hf)(z) = 0$, since $h(x) = (x-z)$, I have considered so then $hf(z) = 0 \Rightarrow$ I will have $L(hf) = (hf)(z) = 0 = \langle hf, g \rangle$ because $L(f) = \langle f, g \rangle$. So, this is true for all f . So, if I replace f by (hf) then also it is true $\Rightarrow 0 = \int_0^1 h(x)f(x) \overline{g(x)} dx \quad \forall f \in V$, where $hf(x) = h(x) f(x)$

\Rightarrow For $f = \overline{h}g$ then we have, $0 = \int_0^1 h(x) \overline{h(x)} g(x) \overline{g(x)} dx \Rightarrow 0 = \int_0^1 |h(x)|^2 |g(x)|^2 dx$
 $|h(x)|$ & $|g(x)|$ is positive function & $|g(x)|$ is non-zero function $\Rightarrow |h(x)||g(x)| = 0 \Rightarrow |g(x)| = 0$ since $|h(x)| \neq 0 \quad \forall x \in (0, 1)$ because $h(x = z) = 0$. So, only possibility is $g(x) = 0$.

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$\Rightarrow g = 0$
 which contradicts our hypothesis that $g \neq 0$
 $\Rightarrow \nexists g \in V$ s.t.
 $\langle f, g \rangle = L(f) = f(z) -$

\checkmark Theorem: Let V be a f.d.i.p space over F . Let T be a linear operator on V . Then there exist an adjoint operator T^* of T such that $\langle T\alpha, \beta \rangle = \langle \alpha, T^*\beta \rangle$ for all $\alpha, \beta \in V$.

Ppf: Consider β be any element of V . Define a fn $\phi: V \rightarrow F$
 $\phi(\alpha) = \langle T\alpha, \beta \rangle \quad \forall \alpha \in V$
 For any $\alpha, \gamma \in V$ & $c \in F$
 $\phi(c\alpha + \gamma) = \langle T(c\alpha + \gamma), \beta \rangle$

So $\Rightarrow g = 0$ which contradict our hypothesis that $g \neq 0 \Rightarrow \nexists g \in V$ such that $\langle f, g \rangle = L(f)z(x)$. So, this example basically shows that if we replace the finite dimensional inner product space by infinite dimensional inner product space. Then the results that is for each linear functionals of our inner product space there will be a fixed vector over the space.

So that the linear functional can be expressed as the $\langle \alpha - \beta \rangle$ the fixed vector. So, here it is not true if it is replaced by in a infinite space. So, we have seen that over a finite dimensional vector space equipped with an inner product, any linear functionals defined on the finite dimensional space can be written as inner product of space vector.

Now, let me use this concept to show the existence of adjoint operator over a finite dimensional inner product space. So, this, let me talk in terms of one small theorems. Theorem: - Let V be a finite dimensional inner product space over the field is F . Let T be a linear operator on V then there exists an adjoint operator T^* of T , such that $\langle T\alpha, \beta \rangle = \langle \alpha, T^*\beta \rangle$ for all $\alpha, \beta \in V$.

So, here I am going to show you that if a space is finite dimensional inner product space and T is a linear operator defined on it. Then exists a linear operator T^* which is called as adjoint of T such that for any two-vector α, β such that $\langle T\alpha, \beta \rangle = \langle \alpha, T^*\beta \rangle$ for all $\alpha, \beta \in V$. Proof:- Consider $\beta \in V$.

Defined a function $\Psi: V \rightarrow F$ as $\Psi(\alpha) = \langle T\alpha, \beta \rangle \forall \alpha \in V$. So, according to this definition, I can see for any $\alpha, \gamma \in V$ and $c \in F$, $\Psi(c\alpha + \gamma) = \langle T(c\alpha + \gamma), \beta \rangle$

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$$\begin{aligned} \Psi(c\alpha + \gamma) &= c\langle T\alpha, \beta \rangle + \langle T\gamma, \beta \rangle \\ &= c\Psi(\alpha) + \Psi(\gamma) \\ \Rightarrow \Psi &\text{ is a linear functional.} \\ \text{So } \exists \beta' \in V \text{ s.t.} & \\ \Psi(\alpha) &= \langle \alpha, \beta' \rangle \text{ ---} \\ \Rightarrow \langle T\alpha, \beta \rangle &= \langle \alpha, \beta' \rangle \text{ ---} \\ \text{Consider a f.m. } T^*: V &\rightarrow V \\ \beta &\rightarrow \beta' \Rightarrow T^*(\beta) = \beta' \text{ ---} \\ \text{Then } T^* &\text{ is a linear operator?} \\ \text{For } \gamma, \alpha, \beta \in V \text{ \& } c \in F, \text{ claim} & \\ T^*(c\gamma + \beta) &= cT^*(\gamma) + T^*(\beta) \text{ ---} \end{aligned}$$

So, according to the definition of inner product, we have $\Psi(c\alpha + \gamma) = c\langle T\alpha, \beta \rangle + \langle T\gamma, \beta \rangle = c\Psi(\alpha) + \Psi(\gamma)$. So, this implies Ψ is a linear functional. So, since, we have considered inner product space is finite dimensional. So, according to our last results.

So, $\exists \beta' \in V$ such that $\Psi(\alpha) = \langle \alpha, \beta' \rangle \Rightarrow \langle T\alpha, \beta \rangle = \langle \alpha, \beta' \rangle$. Now, consider a function $T^*: V \rightarrow V$ s.t. $\beta \rightarrow \beta' \Rightarrow T^*(\beta) = \beta'$. See we have considered here β' as a unique vector because we have already seen for any linear functional or given each linear function.

There exists unique element in V such that linear functional as inner product of that unique vectors. So, this β' is unique. Now, if I consider a function $T^*: V \rightarrow V$ s.t. $\beta \rightarrow \beta' \Rightarrow T^*(\beta) = \beta'$, then T^* is a linear operator how? For $\gamma, \alpha, \beta \in V$ & $c \in F$. I have to show $T^*(c\gamma + \beta) = cT^*(\gamma) + T^*(\beta)$, then only I can say T^* is a linear operator on V .

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According to defⁿ of T^*
as how

$$\langle T\alpha, \beta \rangle = \langle \alpha, \beta' \rangle = \langle \alpha, T^*\beta \rangle \text{ ---}$$

$$\Rightarrow \langle \alpha, T^*(c\gamma + \beta) \rangle = \langle T\alpha, c\gamma + \beta \rangle$$

$$= \bar{c} \langle T\alpha, \gamma \rangle + \langle T\alpha, \beta \rangle$$

$$= \bar{c} \langle \alpha, T^*\gamma \rangle + \langle \alpha, T^*\beta \rangle$$

$$= \langle \alpha, cT^*\gamma \rangle + \langle \alpha, T^*\beta \rangle$$

$$\Rightarrow T^*(c\gamma + \beta) = cT^*\gamma + T^*\beta \text{ ---}$$

$$\Rightarrow T^* \text{ is a linear operator on } V$$

According to definition, we have $\langle T\alpha, \beta \rangle = \langle \alpha, \beta' \rangle = \langle \alpha, T^*\beta \rangle \Rightarrow \langle \alpha, T^*(c\gamma + \beta) \rangle = \langle T\alpha, c\gamma + \beta \rangle = \bar{c} \langle T\alpha, \gamma \rangle + \langle T\alpha, \beta \rangle = \bar{c} \langle \alpha, T^*\gamma \rangle + \langle \alpha, T^*\beta \rangle = \langle \alpha, cT^*\gamma \rangle + \langle \alpha, T^*\beta \rangle \Rightarrow T^*(c\gamma + \beta) = cT^*\gamma + T^*\beta$

So, T^* is a linear operator on V .

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Uniqueness of β' \Rightarrow uniqueness of T^*

Here T^* is called as adjoint of T

✓ Let V be a f.d. i.p. spa. Let $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an orthogonal ordered basis for V . Let T be a linear operator on V . Let $[T]_B = A$. Then (i) $A_{jk} = \langle T\alpha_j, \alpha_k \rangle$ ---

(ii) Also if $[T]_B = B$, then $A_{jk} = \bar{B}_{kj}$ ---

Solⁿ: Given $B = \{\alpha_1, \dots, \alpha_n\}$ an orthogonal ordered basis for V

\Rightarrow For any element $\alpha \in V$

$$\alpha = \sum_{j=1}^n \langle \alpha, \alpha_j \rangle \alpha_j \text{ --- (1)}$$

Given A is the matrix representation of T

Uniqueness of β' \Rightarrow uniqueness of T^* . So, we have seen that over the finite dimensional inner product space. If a linear operator T is defined, one can have another operator T^* defined on the vector space V such that $\langle T\alpha, \beta \rangle = \langle \alpha, T^*\beta \rangle$ Here T^* is called as adjoint of T . So, we have seen existence of adjoint operator of a linear operator defined over a finite dimensional inner product space.

Now, let us see what will be the matrix representations of linear operator T and T^* with respect to some orthogonal order basis on the space. Let V be finite dimensional inner products space. This is an inner products space. Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an orthogonal order basis for V let T be a linear operator on the V . Let $[T]_B = A$.

Then, (i) $A_{jk} = \langle T\alpha_j, \alpha_k \rangle$ (ii) Also $[T]_{B^*} = B$, then $A_{jk} = \overline{B_{kj}}$ Let me quickly solve this problem. Given $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ an orthogonal order basis for V so, this implies for any element $\alpha \in V$, $\alpha = \sum_{j=1}^k \langle \alpha, \alpha_j \rangle \alpha_j \rightarrow (*)$

So, this relation is already known to us. You can of course, quickly check that suppose $\alpha = \sum_{i=1}^k c_i \alpha_i$ taking the concept that $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ an orthogonal basis. It would take inner product then (i) (24:05) be the corresponding c_i . So, anyhow so, we can immediately find out the α can be written like this. Given is the matrix representations of the operator T .

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The slide contains the following handwritten derivations:

$$\Rightarrow T\alpha_j = \sum_{k=1}^n A_{kj} \alpha_k \quad (**)$$

$$\text{Again } T\alpha_j = \sum_{k=1}^n \langle T\alpha_j, \alpha_k \rangle \alpha_k \quad (***)$$

$$\Rightarrow A_{kj} = \langle T\alpha_j, \alpha_k \rangle$$

$$\text{Given } [T^*]_B = B$$

$$\Rightarrow T^*\alpha_j = \sum_{k=1}^n \langle T^*\alpha_j, \alpha_k \rangle \alpha_k$$

$$\Rightarrow B_{kj} = \langle T^*\alpha_j, \alpha_k \rangle = \overline{\langle \alpha_k, T^*\alpha_j \rangle}$$

$$= \overline{\langle T\alpha_k, \alpha_j \rangle}$$

$$\Rightarrow B_{kj} = \overline{A_{jk}} = \overline{A_{jk}}$$

So $\Rightarrow T\alpha_j = \sum_{k=1}^n A_{kj} \alpha_k \rightarrow (**)$. Again, $T\alpha_j = \sum_{k=1}^n \langle T\alpha_j, \alpha_k \rangle \alpha_k \rightarrow (***)$. Compare (**) & (***) equations then $\Rightarrow A_{jk} = \langle T\alpha_j, \alpha_k \rangle$ & given $[T]_{B^*} = B \Rightarrow T^*\alpha_j = \sum \langle T^*\alpha_j, \alpha_k \rangle \alpha_k \Rightarrow B_{kj} = \langle T^*\alpha_j, \alpha_k \rangle = \overline{\langle \alpha_k, T^*\alpha_j \rangle} \Rightarrow B_{kj} = \overline{\langle T\alpha_k, \alpha_j \rangle} = \overline{A_{jk}}$

$\Rightarrow B_{kj} = \overline{A_{jk}}$ so, this is the relation between the matrix representation of the operator T and T star with respect to given orthogonal basis. We will discuss more about this with more examples in our next class.